Geometric Phase in Dissipative System

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Abstract

In non-Hamiltonian dissipative systems that have attracting limit cycles there are phase shifts analogous to the geometric phase of Hamiltonian quantum systems. Here the phase variable is the equal-time parametrization of a limit cycle. The phase shift is calculable when the differential equations of the variables of the systems are known. We study such a system with both numerical and analytical methods and show that the phase shift sometimes, but not always, have a geometric meaning. The computation is obtain by using the IBM SP2 at NCHC.

1. Geometric Phase in a Dissipative System

There is an analogous phenomena to geometric phase in classical dissipative systems [1]. They are dissipative oscillatory systems which have attracting limit cycles. By definition an attracting limit cycle of a dynamical system is such that if the system comes to a point sufficiently near a limit cycle it will relax to the cycle. A limit cycle in a dissipative system is therefore plays the role of a state, or an orbit, in a quantum system. The fast motion of the system around a limit cycle is analogous to the motion of the quantum system caused by the action of the Hamiltonian. A phase variable that equal-time parameterizes the limit cycle corresponds to the phase of a quantum state. Now consider adiabatically moving the classical system through a parameter space. The adiabatic condition prevents large fluctuations from a limit cycle, and ensures that once a system enters a limit cycle, it will always stay close to that limit cycle. This is an exact analogy to the adiabatic condition in a quantum system, which is imposed to ensure that the system stays in one state.

We start with a two-dimensional limit cycle whose first order derivative is given by

$$\dot{\theta} = \Omega\left(\theta, \mu(\epsilon t)\right), \tag{1}$$

where Ω is periodic in θ with periodicity 2π, and μ is a set of slowly varying parameters whose adiabatic limit is $\epsilon \to 0$. The equal-time parametrization of the limit cycle is

$$\varphi = \Phi(\theta, \mu) \equiv \omega(\mu) \int_0^\theta d\theta' \Omega(\theta', \mu)^{-1}. \tag{2}$$

where $\omega(\mu)$ is the inverse period of the cycle when μ is fixed, *i.e.*, $\Phi(2\pi, \mu) = 1$. The integral is an average over the fast motion variable θ. The time derivative of equation (2) is

$$\begin{aligned} \frac{d\varphi}{dt} &= \frac{d\theta}{dt}\partial_\theta \Phi + \frac{d\mu}{dt} \cdot \nabla_\mu \Phi \\ &= \omega(\mu) + \epsilon\dot{\mu} \cdot \nabla_\mu \Phi \end{aligned} \tag{3}$$

The first term of equation (3) is the dynamical phase, and the second term, the one we are interested in, is the rate of change of the phase caused by the adiabatic change of parameters. After completing one closed loop in the parameter space, this phase changes by

$$\Delta\varphi = \epsilon \int_0^\tau dt \dot{\mu}(\epsilon t) \cdot \nabla_\mu \Phi\left(\theta(t), \mu(\epsilon t)\right), \tag{4}$$

where τ is the time needed to complete the loop in the parameter space. The gradient $\nabla_\mu \Phi\left(\theta(t), \mu(\epsilon t)\right)$ is a fast changing function of θ. In the limit $\epsilon \to 0$ we are only interested in slow variations. So we replace it by its angle average:

$$\begin{aligned} \nabla_\mu \Phi\left(\theta(t), \mu(\epsilon t)\right) &\to \omega(\mu) \int_0^{2\pi} \frac{d\theta}{\Omega(\theta, \mu)} \nabla_\mu \Phi(\theta, \mu) \\ &= \langle \Phi'(\theta, \mu) \nabla_\mu \Phi(\theta, \mu) \rangle_\theta, \end{aligned} \tag{5}$$

where $\langle\rangle_\theta$ means integration over θ and the prime on Φ means differentiation with respect to θ. The angle average has the correct normalization because by definition $\omega \int_0^{2\pi} d\theta \Omega^{-1}(\theta) = 1$. Changing the time integration to an integration over the parameters μ, the integrand in (4) takes the form of a one-form,

$$\chi(\mu) \equiv d\mu \cdot \langle \partial_\theta \Phi(\theta, \mu) \nabla_\mu \Phi(\theta, \mu) \rangle, \tag{6}$$

and, in the $\epsilon \to 0$ limit, the phase change becomes a line integral over a closed path C in μ-space,

$$\Delta\varphi(C) = \oint_C \chi. \tag{7}$$

This is transformed by Stokes' theorem to a surface integral,

$$\Delta\varphi(C) = \int_{\partial S=C} d\chi, \tag{8}$$

where the two-form is

$$\begin{aligned} d\chi &= \langle \nabla_\mu \Phi' \times \nabla_\mu \Phi \rangle_\theta \cdot dS \\ &= \int_0^{2\pi} d\theta \, \nabla_\mu \Phi'(\theta) \times \nabla_\mu \Phi(\theta) \cdot dS \end{aligned} \tag{9}$$

This is to be compared with Berry's two-form [2] for a quantum system

$$\begin{aligned} d\chi_{Berry} &= i\langle \nabla_{\mathbf{R}} n| \times |\nabla_{\mathbf{R}} n\rangle \cdot dS \\ &= i \int dq \, \nabla_{\mathbf{R}} \Phi_n^*(q) \times \nabla_{\mathbf{R}} \Phi_n(q) \cdot dS, \end{aligned} \tag{10}$$

where q is the coordinate system in which the representation of the state $|n\rangle$ is the wave-function $\Phi_n(q)$. With the exception that the complex conjugate $i\Phi_n^*(q)$ in the quantum case is replaced by the derivative $\Phi_n'(\theta)$, the similarity is obvious. It should be pointed out that the two-form vanishes when either the complex conjugate or the derivative is not taken.

Now consider the system initially being not exactly on the limit cycle R but is near by, and define the small deviation from the limit cycle as $x = r - R$. Then

$$\dot{\theta} = \Omega(x, \theta, \mu), \tag{11}$$

and the deviation satisfies

$$\dot{x} = -\Lambda f(x, \theta, \mu) - \epsilon\dot{\mu}\nabla_\mu R(\theta, \mu), \tag{12}$$

where $f(0, \theta, \mu) = 0$ and Λ^{-1} is the scale of the relaxation time for returning to the limit cycle. In the limit $\Lambda \to \infty$ the system will fall onto the limit cycle very rapidly. For large Λ one can expand x in Λ^{-1}:

$$x(t, \Lambda) = \sum_{m=1}^{\infty} x_m(t)\lambda^{-m} + \xi(t, \Lambda). \tag{13}$$

The second term on the right-hand-side is a transient term dependent on the initial condition and decays exponentially as $\exp(-a\Lambda t)$ for some constant $a > 0$. Since we are considering an indefinitely long time, this term can be neglected. Substituting (13) into (12), Taylor expand f to $O(\Lambda^{-1})$ and neglect all terms of order ϵ^2, x^2, ϵx or higher, we obtain

$$\begin{aligned} \frac{dx}{dt} &= \sum_{m=1}^{\infty} \Lambda^{-m}\frac{dx_m}{dt} = -\Lambda x \partial_x f - \epsilon\dot{\mu} \cdot \nabla_\mu R \\ &= -\sum_{m=1}^{\infty} x_m \Lambda^{-m+1} f' - \epsilon\dot{\mu} \cdot \nabla_\mu R, \end{aligned} \tag{14}$$

Note that $dx_m/dt = (\epsilon\dot{\mu} \cdot \nabla_\mu + \dot{\theta}\partial_\theta)x_m$ and $\Lambda^{-m} \cdot \epsilon$ also should be neglected for any $m \geq 1$. Equating the coefficients of terms of the same power in Λ^{-1}, we get

$$\begin{aligned} x_1 &= -\left(\frac{\partial f}{\partial x}\right)^{-1} \epsilon\dot{\mu} \cdot \nabla_\mu R(\theta, \mu), \\ \Omega(0, \theta, \mu)\frac{\partial x_m}{\partial \theta} &= -x_{m+1}\frac{\partial f}{\partial x}(0, \theta, \mu), \end{aligned} \tag{15}$$

for $m \geq 1$. Since each term in the series is proportional to $\epsilon\dot{\mu}$, x has the form

$$x = -\epsilon\dot{\mu} \cdot \zeta(\theta, \mu, \Lambda). \tag{16}$$

Taylor expanding the new Ω in (11) in x, we have

$$\begin{aligned} \omega^{-1}(x, \mu) &= \int_0^{2\pi} \frac{d\theta}{\Omega(x, \theta, \mu)} \\ &\approx \omega^{-1}(\mu) \left(1 - \omega(\mu)x \right. \\ &\qquad \left. \int_0^{2\pi} \frac{d\theta}{\Omega(\theta, \mu)} \partial_x \ln \Omega(0, \theta, \mu)\right). \end{aligned} \tag{17}$$

Hence

$$\omega(x, \mu) \approx \omega(\mu) \left(1 + \omega(\mu)x \int_0^{2\pi} \frac{d\theta}{\Omega(\theta, \mu)} \partial_x \ln \Omega(0, \theta, \mu)\right), \tag{18}$$

and (3) becomes

$$\frac{d\varphi}{dt} = \omega(\mu) + \epsilon\dot{\mu}\left(-\omega(\mu)^2\zeta \int_0^{2\pi} \frac{d\theta}{\Omega}\partial_x \ln \Omega + \nabla_\mu \Phi\right), \tag{19}$$

which adds an additional term to the one-form

$$\begin{aligned} \chi &= d\mu \cdot \omega(\mu) \int_0^{2\pi} \frac{d\theta}{\Omega} \left[\nabla_\mu \Phi(\theta, \mu) \right. \\ &\qquad \left. -\omega(\mu)^2\zeta \int_0^{2\pi} \frac{d\theta}{\Omega}\partial_x \ln \Omega\right]. \end{aligned} \tag{20}$$

Since there is no need to the angle average twice, the one-form can finally be written as

$$\begin{aligned} \chi &= d\mu \cdot \langle \Phi'(\theta, \mu) \left[\nabla_\mu \Phi(\theta, \mu) - \omega(\mu)\zeta\partial_x \ln \Omega\right]\rangle_\theta \\ &\equiv \chi_1 + \chi_2, \end{aligned} \tag{21}$$

where χ_1 is the part of χ independent of x and χ_2 is the part that does, and Φ, ζ and Ω are all evaluated at $x = 0$. Evidently (8) still holds.

2. An Example

Here we consider a case where it is possible to derive an approximate analytic expression for the phase shift; the result is proportional to the area enclosed by the closed path. Then we compare it against the numerically computed phase shift for several geometrically distinct paths.

2.1. Approximate analytic expression

Consider a dissipative system with one attracting limit cycle whose evolution is determined by the set of equations:

$$\begin{aligned}\dot{\theta} &= r, \\ \dot{r} &= r\tfrac{\partial R}{\partial \theta} + (R - r),\end{aligned} \tag{22}$$

where the pair of parameters is $\mu = (\rho, \psi)$ with $0 \leq \rho < 1$ and the limit cycle is an ellipse,

$$R = 1/\left(1 + \rho\cos(\theta + \psi)\right). \tag{23}$$

When the system is initially on the limit cycle, $\dot{\theta} \simeq R$ and the period ω^{-1} is

$$\omega^{-1} = \int_0^{2\pi} \frac{d\theta}{\dot{R}} = 2\pi, \tag{24}$$

which is independent of the parameters, and the function Φ is

$$\Phi = \frac{1}{2\pi}\int_\theta^{2\pi} \left(1 + \rho\cos(\theta + \psi)\right) = \frac{1}{2\pi}\left(\theta + \rho\sin(\theta + \psi)\right). \tag{25}$$

From (21) the first part of the one-form χ is,

$$\begin{aligned}\chi_1 &= d\mu \cdot \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{R}\left(\sin(\theta + \psi)\hat{\rho} + \cos(\theta + \psi)\hat{\psi}\right) \\ &= \frac{1}{4\pi}\rho^2 d\psi,\end{aligned} \tag{26}$$

and phase shift produced by it is

$$\Delta\varphi_1 = \int_0^{2\pi} d\psi \frac{\rho^2}{4\pi} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\rho^2 d\psi. \tag{27}$$

The phase shift is the area enclosed by the closed path in the parameter space divided by 2π.

Now consider fluctuations about the limit cycle. Because $x \equiv r - R$, we can get the equation for x from the equation of r

$$\dot{r} = \dot{R} + \dot{x} = \dot{\theta}\frac{\partial R}{\partial \theta} - x, \tag{28}$$

and re-express (22) as equations of θ and x,

$$\begin{aligned}\dot{\theta} &= R + x, \\ \dot{x} &= -x - \epsilon\dot{\mu}\nabla_\mu R.\end{aligned} \tag{29}$$

Change the unit of time to $t' = t/\tau$, then $\dot{x}$ becomes

$$\frac{dx}{dt'} = -\tau x - \dot{\mu}\nabla_\mu R. \tag{30}$$

A comparison with (14) yields

$$f(x, \theta, \mu) = x, \qquad \Lambda = \tau, \qquad \partial f/\partial x = 1. \tag{31}$$

The solution for (15) on the series expansion of x is

$$\begin{aligned}x_1 &= -\dot{\mu} \cdot \nabla_\mu R, \\ x_2 &= \dot{\mu} \cdot R\partial_\theta \nabla_\mu R, \\ &\cdots\end{aligned} \tag{32}$$

Thus,

$$x = \sum_{m=1}^{\infty} x_m \tau^{-m} = -\frac{1}{\tau}\dot{\mu}\zeta, \tag{33}$$

where

$$\zeta = \nabla_\mu R + \frac{1}{\tau}R\partial_\theta \nabla_\mu R + O(\tau^{-2}). \tag{34}$$

We can then obtain χ_2 and $\Delta\phi_2$.

2.2. Numerical results for some paths

As a first and simple example, we keep $\rho = \rho_0$ as a constant and move the system along a circle in the parameter space at constant speed so that $\psi = 2\pi t/\tau$. Then from equation (27),

$$\Delta\varphi_1 = \frac{1}{2}\rho_0^2. \tag{35}$$

To compute the χ_2 we have to $O(\tau^{-1})$

$$\zeta = R^2\sin(\theta + \psi) - \frac{1}{\tau}\left(2R^4\rho\sin^2(\theta + \psi) + R^3\cos(\theta + \psi)\right). \tag{36}$$

Then from (21)

$$\begin{aligned}\chi_2 &= -d\mu \cdot \omega(\mu)\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{R}\zeta\partial_x \ln\Omega \\ &\approx -d\mu \cdot \frac{1}{4\pi^2}\int_0^{2\pi} \frac{d\theta}{R}\frac{1}{R}\Big[R^2\sin(\theta + \psi) \\ &\qquad -\frac{1}{\tau}\left(2R^4\rho_0\sin^2(\theta + \psi) + R^3\cos(\theta + \psi)\right)\Big] \\ &= -\frac{1}{2\pi\tau}d\psi,\end{aligned} \tag{37}$$

so $\Delta\varphi_2 = -1/\tau$. The total phase shift is therefore

$$\Delta\varphi = \frac{1}{2}\rho_0^2 - \frac{1}{\tau} + O\left(\frac{1}{\tau^2}\right), \qquad \lim_{\tau\to\infty} \Delta\varphi = \frac{1}{2}\rho_0^2. \tag{38}$$

In the case $\psi = -2\pi t/\tau$, the system is moved along the circle in the parameter space in the opposite direction. Then $\Delta\varphi = -\frac{1}{2}\rho_0^2 + 1/\tau + O(\frac{1}{\tau^2})$. These results are shown in Figure 1.

In general the phase shift may not be calculable analytically. we have solved the the set of differential equations (22) by numerical integration using the fourth-order Runge-Kutta method [4] for several cases.

In practice we compute $\theta(t)$ at any t. In particular we get $\varphi(\tau) = \theta(\tau)$. The dynamical phase denoted by θ_0 may be obtained similarly by keeping ψ a constant; it should be and is equal to $2\pi\omega\tau$. Then the phase shift is

$$\Delta\varphi = \frac{1}{2\pi}\left(\theta(\tau) - \theta_0(\tau)\right). \tag{39}$$

Where the normalization is to conform with the definition of φ (equation (2)), which is an angle divided by 2π. Because $\varphi = \int d\chi + O(\epsilon)$, we only compare numerically and analytically computed results when ϵ is small (*i.e.*, when τ is large). The results are shown in Table 1 and Figure 1 and 2. The solid lines is the $\Delta\phi$ for $\psi = 2\pi t/\tau$ and the dashed line is the $-\Delta\phi$ for $\psi = -2\pi t/\tau$ The analytical result appears to have errors of the order of $10^{-2} \sim 10^{-1}$.

In the second case we still move the system around a circle but this time the center of the circle is not on the origin. The function of the parameters are $\rho = \rho_0 \sin(2\pi t/\tau)$ and $\psi = 2\pi t/\tau$. By this functions, the system will move around the circle twice. The analytical result when $\tau \to \infty$ is

$$\varphi = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\rho_0^2 \sin^2\psi d\psi = \frac{1}{2\pi}(\pi\frac{\rho_0^2}{2}). \tag{40}$$

The figure of the path and the numerical results are shown in Figure 3 and 4. The errors of the analytical results are of the order of 10^{-1}.

In the third case, the functions of the parameters are $\rho = \rho_0 \sin(2\pi t/\tau)$ and $\psi = \psi_0 \cos(2\pi/\tau)$. The figure of the path is shown in Figure 5. In this case the total area enclosed by the path is equal to zero, and the numerical result also approaches zero ,as shown in Figure 6.

In the fourth case, the system move along an eight-form path, as shown in Figure 7. The total area enclosed by the path is also equal to zero. The numerical result is approaches to zero, too, as shown in Figure 8. Figure 10 is the numerical result when the closed path is a square, as shown in Figure 9. Here the parameter $\psi = 2\pi t/\tau$. The analytic result for $\tau \to \infty$ is $\Delta\psi = 4\rho_0^2/2\pi$. The error in this case is of the order of 10^{-2}. All these results are obtained from IBM SP2 machine.

In summary, we can say that the phase shift in a dissipative system with limit cycles is geometric, because $\Delta\phi$ approaches some constant at the adiabatic limit. The phase shift will not increase with the time limit cycles is geometric, because $\Delta\phi$ approaches some constant at the adiabatic limit. The phase shift will not increase with the time to complete the circuit in the parameter space. In the example shown in section 4, the geometric property of the phase shift is that it is proportional to the area enclosed by the circuit in the parameter space. The numerical results also agree with this property. But this is not a general rule for the phase shift. The geometric properties of the phase shift in dissipative systems depend on the case given, not like the quantum systems which have a general rules for the phase shift. We can obtain the geometric properties of the phase shift in other cases by the formulae given in section 3.

References

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Table 1. Analytical and numerical results of the circle path.

ρ_0	τ	analytical reults	numerical results	error %
0.2	10^4	0.1257	0.1568	19.878
0.3	10^4	0.2827	0.3260	13.256
0.4	10^4	0.5027	0.5410	7.089
0.5	10^4	0.7854	0.8008	1.919
0.6	10^4	1.1310	1.1156	1.382
0.7	10^4	1.5394	1.5224	1.118
0.8	10^4	2.0106	2.1951	8.405

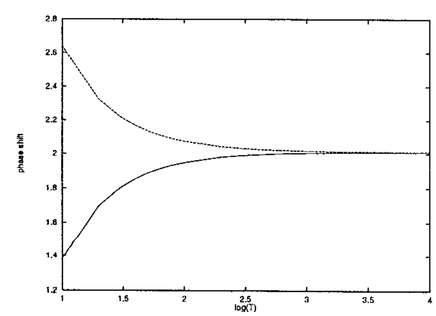

Figure 1. Analytical results of example 1. Here $\rho_0 = 0.8$.

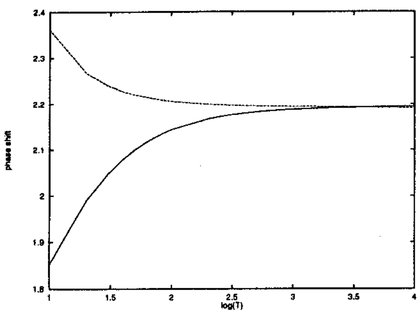

Figure 2. Numerical results of example 1. Here $\rho_0 = 0.8$.

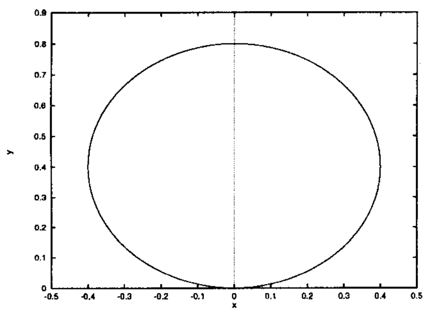

Figure 3. The path in the parameter space of exsample 2.

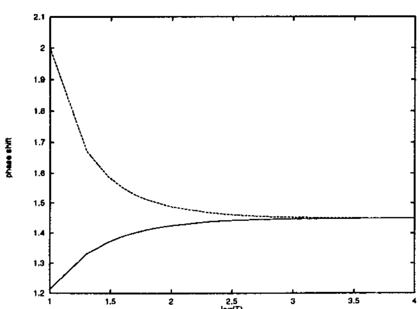

Figure 4. $\Delta\phi$ for moving around the path shown in Fig.3.

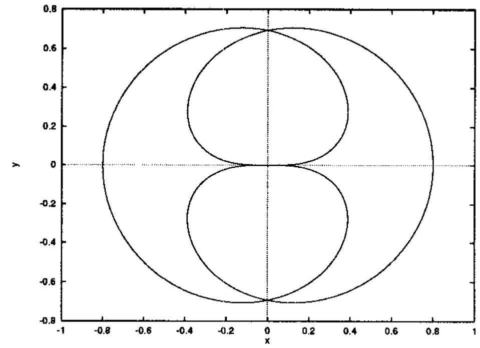

Figure 5. The path in the parameter space of example 3.

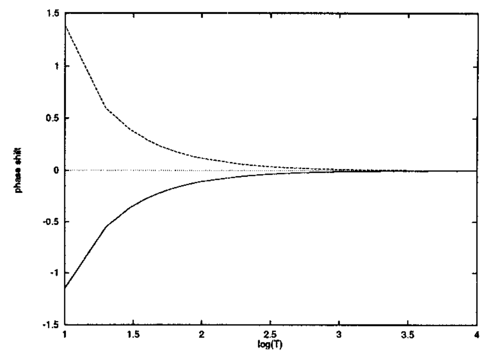

Figure 6. $\Delta\phi$ for moving around the path shown in Fig.5.

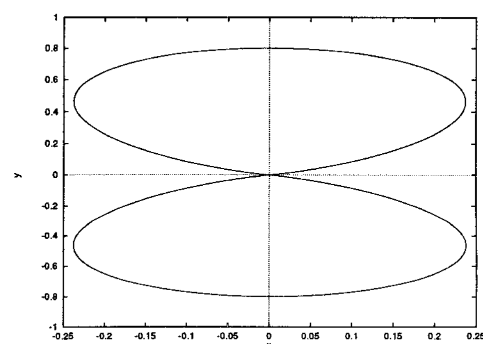

Figure 7. The path in the parameter space of example 4.

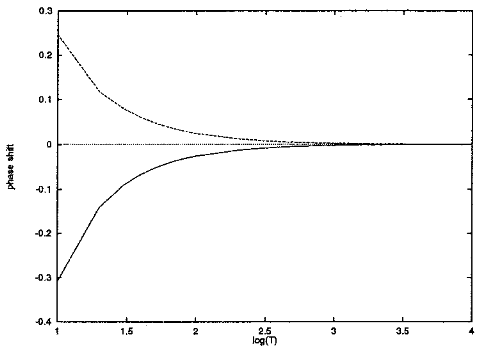

Figure 8. $\Delta\phi$ **for moving around the path shown in Fig.7.**

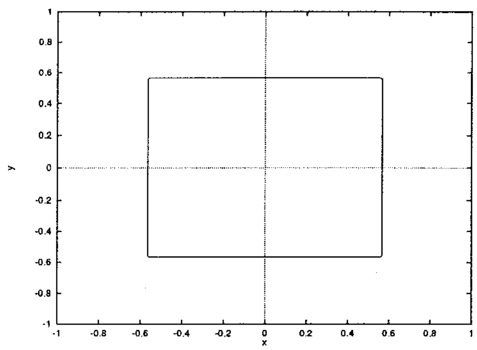

Figure 9. A square path in the parameter space of example 5.

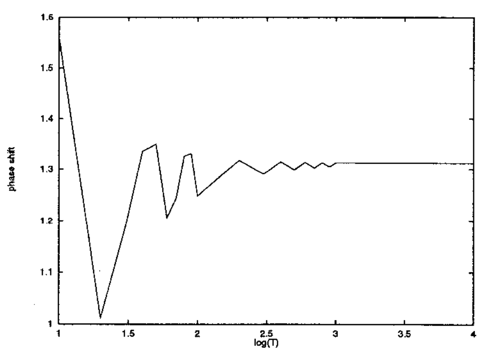

Figure 10. $\Delta\phi$ **for moving around the path shown in Fig.9.**