Colored solutions of the Yang–Baxter equation from representations of $U_q gl(2)$

Pi-Gang Luan[a]
Department of Physics, National Central University, Chung-li, Taiwan, Republic of China

H. C. Lee[b]
Department of Physics, National Central University, Chung-li, Taiwan, Republic of China and National Center for Theoretical Sciences, P.O. Box 2-131, Hsinchu, Taiwan, Republic of China

R. B. Zhang[c]
Department of Mathematics, University of Queensland, Brisbane, Queensland 4072, Australia

(Received 6 December 1999; accepted for publication 18 May 2000)

We study the Hopf algebra structure and the highest weight representation of a multiparameter version of $U_q gl(2)$. The Hopf algebra maps of this algebra are explicitly given. We show that the multiparameter universal $\mathcal{R}$ matrix can be constructed directly as a quantum double intertwiner without using Reshetikhin's twisting transformation. We find there are two types highest weight representations for this algebra: type a corresponds to the qeneric q and type b corresponds to the case that q is a root of unity. When applying the representation theory to the multiparameter universal $\mathcal{R}$ matrix, both standard and nonstandard colored solutions of the Yang–Baxter equation are obtained. © *2000 American Institute of Physics.* [S0022-2488(00)02409-9]

I. INTRODUCTION

As is well known, the Yang–Baxter equation (YBE)[1,2] plays an essential role in the study of quantum groups (QG) and quantum algebras (QA),[3–8] integrable models,[9–12] as well as in the construction of knot or link invariants.[13–19] For instance, in the Faddeev–Reshetikhin–Takhtajan (FRT) approach[5–7] to construct quantum groups or quantum algebras, one has to find an R matrix, which is a matrix solution of the YBE,[2] then, using this R matrix as input, substitute it into the RTT or RLL relations to get the quantum group or quantum algebra as output.

There are various methods to find the appropriate R matrix. One way is to borrow an $R(u)$ matrix from the integrable model[2] and then take an appropriate limit to remove the spectral parameter u. The second method is to solve the matrix YBE directly.[19–21] In this approach one usually assumes an R with prescribed nonzero elements, and impose some restrictions on them to find a class of solutions. Some R matrices obtained in this way have unexpected interesting features, so a number of authors call them ''nonstandard'' solutions.[22–26]

Many known quantum algebras belong to the category of quasitriangular Hopf algebras (QTHA).[8] This observation provides us an alternative approach to find R matrices. When applying representation theory to the universal $\mathcal{R}$ matrix[8] of a QTHA, the desired R matrix is obtained. (We denote the universal algebraic solution of YBE by $\mathcal{R}$ and the matrix solution by R.) To get more interesting solutions, people also try various methods to additional parameters into the R matrix.[27,28] This has led to the study of q-boson realizations[29–33] with q being a root of unity and the multiparameter deformations[34–39] of Hopf algebras. These solutions are sometimes called

[a]Electronic mail: luan@fermi.phy.ncu.edu.tw
[b]Electronic mail: hclee@sansan.phy.ncu.edu.tw
[c]Electronic mail: rzhang@maths.uq.edu.au

''colored'' solutions.[18,38] Although the q-boson realization method is very powerful in constructing representations of quantum groups or quantum algebras, it tends to obscure the structure of the Hopf algebra.

In this article we study $U_q gl(2)$. We show that the appearance of the commuting element J makes it possible to introduce additional parameters t, u and v, and hence yields a multiparameter version of Hopf maps and a multiparameter universal $\mathcal{R}$ matrix. We then explain how to get the same $\mathcal{R}$ from quantum double construction. In this way the Hopf algebra structure is preserved and emphasized. We also compare our results with those obtained via Reshetikhin's twisting transformation.

We consider only the highest weight representations of $U_q gl(2)$. Under the finite-dimension restriction, two categories of representation appear automatically. When applying this representation theory to $\mathcal{R}$, the standard and nonstandard colored solutions are obtained and these are consistent with those published in the literature.

This article is organized as follows: In Sec. II, we review some basic definitions and properties of Hopf algebras, quasitriangular Hopf algebras and quantum double. In Sec. III, different choices of *coproduct*, *antipode* and universal $\mathcal{R}$ matrices are given. We also briefly discuss quantum double construction and its relation with the multiparameter version of $U_q gl(2)$. In Sec. IV, we compare our results to those obtained from Reshetikhin's twisting transformation.[35] In Sec. V, the highest weight representations are studied and applied to $\mathcal{R}$ to obtain matrix solutions R. In Sec. VI, colored solutions are obtained and compared to the literature. Section VII contains concluding remarks.

II. HOPF ALGEBRAS, QUASI-TRIANGULAR HOPF ALGEBRAS AND QUANTUM DOUBLE

In this section we give a brief review of some definitions and properties of Hopf algebras (HA) and quasi-triangular Hopf algebras (QTHA) and their relations to the notion of quantum double (QD).[8] These ideas will be used in our latter discussions of the multiparameter $U_q gl(2)$.

A. Hopf algebras

A Hopf algebra is an associative algebra A with five basic maps (in this article, we call them Hopf maps), namely, four homomorphisms: $m:A\otimes A\to A$ (*multiplication*), $\Delta:A\to A\otimes A$ (*coproduct*), $\eta:C\to A$ (*inclusion*), $\varepsilon:A\to C$ (*counit*), and one antihomomorphism: $S:A\to A$ (*antipode*). They satisfy the following relations for any $a\in A$:

$$
\begin{gathered}
(\Delta\otimes \mathrm{id})\Delta(a)=(\mathrm{id}\otimes\Delta)\Delta(a),\\
(\varepsilon\otimes \mathrm{id})\Delta(a)=(\mathrm{id}\otimes\varepsilon)\Delta(a)=\mathrm{id}(a)=a, \qquad (2.1)\\
m(S\otimes \mathrm{id})\Delta(a)=m(\mathrm{id}\otimes S)\Delta(a)=\eta\circ\varepsilon(a)=\varepsilon(a)1,
\end{gathered}
$$

where id is the *identity map*. To be more precise, we use the notation $(A,m,\Delta,\eta,\varepsilon,S)$ instead of A to denote a Hopf algebra. The following proposition is apparent:

Proposition II.1: The algebra $(A,m,\Delta',\eta,\varepsilon,S^{-1})$ *is also a Hopf algebra.*

Here Δ' denotes the opposite coproduct, which maps any $a\in A$ to $A\otimes A$ as

$$\Delta'(a)=\sigma\circ\Delta(a)=\sum_i c_i\otimes b_i \quad \text{if } \Delta(a)=\sum_i b_i\otimes c_i, \qquad (2.2)$$

and S^{-1} is defined as the inverse of S:

$$S(S^{-1}(a))=S^{-1}(S(a))=a. \qquad (2.3)$$

B. Quasitriangular Hopf algebras

A quasitriangular Hopf algebra (QTHA) is a Hopf algebra equipped with an element $\mathcal{R}\in A\otimes A$ which is the solution of the algebraic version of YBE. We start with the definition.

Definition II.1: Let $\mathcal{A}=(A,m,\Delta,\eta,\epsilon,S)$ be a Hopf algebra and $\mathcal{R}$ (intertwiner) be an invertible element in $A\otimes A$. Then the pair $(\mathcal{A},\mathcal{R})$ is called a QTHA if for any $a\in A$ we have

(i) $\mathcal{R}\Delta(a)=\Delta'(a)\mathcal{R}$,
(ii) $(\Delta\otimes\mathrm{id})\mathcal{R}=\mathcal{R}_{13}\mathcal{R}_{23}$,
(iii) $(\mathrm{id}\otimes\Delta)\mathcal{R}=\mathcal{R}_{13}\mathcal{R}_{12}$.

Here, for example, $\mathcal{R}_{13}$ lives in the first and third sections in $A\times A\times A$.

By definition, the three relations are satisfied:

$$\begin{gathered}\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}=\mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12},\\ (S\otimes\mathrm{id})\mathcal{R}=(\mathrm{id}\otimes S^{-1})\mathcal{R}=\mathcal{R}^{-1},\\ (\varepsilon\otimes\mathrm{id})\mathcal{R}=(\mathrm{id}\otimes\varepsilon)\mathcal{R}=1.\end{gathered}\tag{2.4}$$

The first line is the Yang–Baxter equation.

As in the case of Hopf algebras, we denote $(A,\mathcal{R},m,\Delta,\eta,\varepsilon,S)$ as a QTHA. From (i) of Definition II.1, we immediately find

$$\mathcal{R}\Delta(a)=\Delta'(a)\mathcal{R},\quad (\sigma\circ\mathcal{R})\Delta'(a)=\Delta(a)(\sigma\circ\mathcal{R}),$$
$$\mathcal{R}^{-1}\Delta'(a)=\Delta(a)\mathcal{R}^{-1},\quad (\sigma\circ\mathcal{R}^{-1})\Delta(a)=\Delta'(a)(\sigma\circ\mathcal{R}^{-1}).$$

Define $\mathcal{R}^{(+)}=\sigma\circ\mathcal{R}$, $\mathcal{R}^{(-)}=\mathcal{R}^{-1}$ and $\overline{\mathcal{R}}=\sigma\circ\mathcal{R}^{-1}$, then

$$\mathcal{R}\Delta=\Delta'\mathcal{R},\quad \overline{\mathcal{R}}\Delta=\Delta'\overline{\mathcal{R}}.\tag{2.5a}$$

Also, writing Δ' as $\overline{\Delta}$, then

$$\mathcal{R}^{(+)}\overline{\Delta}=\overline{\Delta}'\mathcal{R}^{(+)},\quad \mathcal{R}^{(-)}\overline{\Delta}=\overline{\Delta}'\mathcal{R}^{(-)}.\tag{2.5b}$$

These observations lead to the following result:

Proposition II.2: If $(A,\mathcal{R},m,\Delta,\varepsilon,S,\eta)$ is a QTHA, then $(A,\overline{\mathcal{R}},m,\Delta,\varepsilon,S,\eta)$, $(A,\mathcal{R}^{(+)},m,\Delta',\varepsilon,S^{-1},\eta)$ and $(A,\mathcal{R}^{(-)},m,\Delta',\varepsilon,S^{-1},\eta)$ are all QTHAs.

This can be easily proved by using Definition II.1 and Eq. (2.1). It tells us that for a pair (Δ,S), there are two universal $\mathcal{R}$ matrices: $\mathcal{R}$ and $\overline{\mathcal{R}}=\sigma\circ\mathcal{R}^{-1}$. Both can be used as intertwiner in a QTHA. Now let us turn to the discussion of *quantum double*.[8]

C. Quantum double

Suppose we have a Hopf algebra A that is spanned by basis $\{e_i\}$. By introducing a nondegenerate bilinear form $\langle\,,\rangle$, we can define A's dual algebra A^o that is spanned by $\{e^i\}$; here $\langle e^i,e_j\rangle=\delta^i_j$. Then all the Hopf maps of A^o can be defined in terms of $\langle\,,\rangle$. Introduce the *intertwiner*,

$$\mathcal{R}=\sum_i e_i\otimes e^i.\tag{2.6}$$

The commutation relations between A and A^o can be established via the relation

$$\mathcal{R}\Delta(a)=\Delta'(a)\mathcal{R},\quad \text{for}\quad a\in A \quad \text{or}\quad A^o,$$

which tells us how to expand an e^ie_j type product as a sum of e_ie^j type products. Choosing $\{e_ie^j\}$ as the basis, one can ''combine'' A and A^o to form an enlarged algebra $D(A)$, the *quantum double* of A, and treat A or A^o as its subalgebra. Then $D(A)$ can be proved to be a QTHA equipped with $\mathcal{R}=\Sigma_ie_i\otimes e^i$ as its intertwiner (universal $\mathcal{R}$ matrix). In other words, a QTHA is a quantum double of its subalgebra. In the next section, we will show that the $U_qgl(2)$ is a quantum double as well as a QTHA.

III. UNIVERSAL $\mathcal{R}$ MATRIX OF $U_qgl(2)$

We define our version of $U_qgl(2)$ algebra as a multiparameter QTHA generated by (H,J,X^+,X^-) with the commutation relations

$$\begin{aligned}
&[J,H]=[J,X^\pm]=0,\\
&[H,X^\pm]=\pm 2X^\pm, \qquad (3.1)\\
&[X^+,X^-]=\frac{q^Ht^{-J}-q^{-H}t^J}{q-q^{-1}};
\end{aligned}$$

coproduct, antipode and counit,

$$\text{coproduct:}\begin{cases}\Delta(H)=H\otimes 1+1\otimes H,\quad \Delta(J)=J\otimes 1+1\otimes J,\\ \Delta(X^+)=q^{-(1/2)H}(utv)^{(1/2)J}\otimes X^+ + X^+\otimes q^{(1/2)H}(utv^{-1})^{-(1/2)J},\\ \Delta(X^-)=q^{-(1/2)H}(u^{-1}tv^{-1})^{(1/2)J}\otimes X^- + X^-\otimes q^{(1/2)H}(u^{-1}tv)^{-(1/2)J},\end{cases} \qquad (3.2)$$

$$\text{antipode:}\quad S(H)=-H,\quad S(J)=-J,\quad S(X^\pm)=-q^{\pm 1}v^{\mp J}X^\pm, \qquad (3.3)$$

$$\text{counit:}\quad \varepsilon(H)=\varepsilon(J)=\varepsilon(X^\pm)=0; \qquad (3.4)$$

and the universal $\mathcal{R}$ matrix given by

$$\mathcal{R}=\mathcal{R}_0\sum_{n=0}^{\infty}\frac{(1-q^2)^n}{\{n\}_{q^2}!}q^{n(n-1)}q^{(n/2)(H\otimes 1-1\otimes H)}((uvt^{-1})^{(1/2)J}X^-)^n\otimes((uv^{-1}t)^{(1/2)J}X^+)^n, \qquad (3.5)$$

where

$$\mathcal{R}_0=q^{-(1/2)H\otimes H}t^{(1/2)(H\otimes J+J\otimes H)}u^{(1/2)(H\otimes J-J\otimes H)}, \qquad (3.6)$$

t, u and v are arbitrary parameters, and $\{n\}_{q^2}$ and $\{n\}_{q^2}!$ are defined as

$$\begin{aligned}
&\{n\}_{q^2}=\frac{1-q^{2n}}{1-q^2}=q^{n-1}[n]_q,\\
&\{n\}_{q^2}!=\prod_{j=1}^{n}\{j\}_{q^2}=q^{(1/2)n(n-1)}[n]_q!,
\end{aligned} \qquad (3.7)$$

with $\{0\}_{q^2}!=[0]_q!=1$. Note that since the generator J commutes with each element in $U_qgl(2)$, different expressions of Hopf maps (i.e., *multiplication*, *coproduct*, *antipode* and *counit*) are possible. If we replace H by $H'=H-cJ$ and define $t'=q^{-c}t$ (here c is an arbitrary constant), then a new Hopf map is obtained by the replacement

$$H\rightarrow H',\quad t\rightarrow t'.$$

The parameter t can even be made to disappear when one defines $q^{H}t^{-J}=q^{\tilde{H}}$ and replaces the generator H by $\tilde{H}$. For reasons that will become clear we shall retain the parameter t. In addition to t, two more parameters, u and v, appear in the expressions for $\Delta(X^{\pm})$, although they do not explicitly appear in the commutation relations (3.1). It is important to note that t, u and v are all *arbitrary* parameters. Knowing this allows one to see that many so-called ''multiparameter deformations'' of $U_q gl(2)$ are in fact different expressions of $U_q gl(2)$. We will discuss this point further in the next section.

Consider the transformation

$$\tilde{X}^{\pm}=v^{\mp 1/2\,J}X^{\pm}, \tag{3.8}$$

under which the expressions for Δ and S on $X^{\pm}$ become

$$\Delta(\tilde{X}^{\pm})=(q^{-(1/2)\,H}t^{(1/2)\,J})u^{\pm(1/2)\,J}\otimes\tilde{X}^{\pm}+\tilde{X}^{\pm}\otimes(q^{(1/2)\,H}t^{-(1/2)\,J})u^{\mp(1/2)\,J}, \tag{3.9}$$

$$S(\tilde{X}^{\pm})=-q^{(1/2)\,H}\tilde{X}^{\pm}q^{-(1/2)\,H}=-q\tilde{X}^{\pm}, \tag{3.10}$$

and the commutation relations (3.1) preserve their form (with the replacement $X^{\pm}\rightarrow\tilde{X}^{\pm}$). Moreover, the universal $\mathcal{R}$ matrix now becomes

$$\mathcal{R}=\mathcal{R}_0\sum_{n=0}^{\infty}\frac{(1-q^2)^n}{[n]_q!}q^{-(1/2)\,n(n-1)}((ut^{-1})^{(1/2)\,J}q^{(1/2)\,H}\tilde{X}^{-})^n\otimes((ut)^{(1/2)\,J}q^{-(1/2)\,H}\tilde{X}^{+})^n. \tag{3.11}$$

In the following, we shall use $\tilde{X}^{\pm}$ instead of $X^{\pm}$ as generators.

As stated in the last section, corresponding to the same pair (Δ,S), there is another appropriate universal $\mathcal{R}$ matrix that shares the same Hopf algebra structure:

$$\bar{\mathcal{R}}=\bar{\mathcal{R}}_0\sum_{n=0}^{\infty}\frac{(1-q^{-2})^n}{[n]_q!}q^{(1/2)\,n(n-1)}((ut)^{-(1/2)\,J}q^{(1/2)\,H}\tilde{X}^{+})^n\otimes((u^{-1}t)^{(1/2)\,J}q^{-(1/2)\,H}\tilde{X}^{-})^n \tag{3.12}$$

with

$$\bar{\mathcal{R}}_0=\sigma\circ\mathcal{R}_0^{-1}=q^{(1/2)\,H\otimes H}t^{-(1/2)(H\otimes J+J\otimes H)}u^{(1/2)(H\otimes J-J\otimes H)}. \tag{3.13}$$

Similarly, if we choose $\bar{\Delta}=\Delta'$ and $\bar{S}=S^{-1}$ as another choice of coproduct and antipode, then for the pair $(\bar{\Delta},\bar{S})$, we have the other two universal $\mathcal{R}$ matrices, $\mathcal{R}^{(+)}$ and $\mathcal{R}^{(-)}$:

$$\mathcal{R}^{(+)}=\mathcal{R}_0^{(+)}\sum_{n=0}^{\infty}\frac{(1-q^2)^n}{[n]_q!}q^{-(1/2)\,n(n-1)}((ut)^{(1/2)\,J}q^{-(1/2)\,H}\tilde{X}^{+})^n\otimes((ut^{-1})^{(1/2)\,J}q^{(1/2)\,H}\tilde{X}^{-})^n,$$

$$\mathcal{R}_0^{(+)}=\sigma\circ\mathcal{R}_0=q^{-(1/2)\,H\otimes H}t^{1/2(H\otimes J+J\otimes H)}u^{-(1/2)(H\otimes J-J\otimes H)}, \tag{3.14}$$

and

$$\mathcal{R}^{(-)}=\mathcal{R}_0^{(-)}\sum_{n=0}^{\infty}\frac{(1-q^{-2})^n}{[n]_q!}q^{(1/2)\,n(n-1)}((u^{-1}t)^{(1/2)\,J})q^{-(1/2)\,H}\tilde{X}^{-})^n\otimes((ut)^{-(1/2)\,J}q^{(1/2)\,H}\tilde{X}^{+})^n,$$

$$\mathcal{R}_0^{(-)}=\mathcal{R}_0^{-1}=q^{(1/2)\,H\otimes H}t^{-(1/2)(H\otimes J+J\otimes H)}u^{-(1/2)(H\otimes J-J\otimes H)}. \tag{3.15}$$

These universal $\mathcal{R}$ matrices can be compared to the literature.[27,28,38,39] However, since different authors adopt different conventions in the definition of Δ and S, one has to properly choose one universal $\mathcal{R}$ matrix from the set $\{\mathcal{R}, \overline{\mathcal{R}}, \mathcal{R}^{(+)}, \mathcal{R}^{(-)}\}$ to make the comparison.

If we define $q^{\alpha_1}=u^{-1}t$ and $q^{\alpha_2}=ut$, and use $H_1=H-\alpha_1 J$ and $H_2=H-\alpha_2 J$ as generators instead of H and J, then the form of the universal $\mathcal{R}$ matrix becomes (here a trivial commuting element $-\frac{1}{2}\alpha_1\alpha_2 J\otimes J$ is added to the exponent of $\mathcal{R}_0$):

$$\mathcal{R}=q^{-(1/2)H_1\otimes H_2}\sum_{n=0}^{\infty}\frac{(1-q^2)^n}{[n]_q!}q^{-(1/2)n(n-1)}(q^{(1/2)H_1}\tilde{X}^-)^n\otimes(q^{-(1/2)H_2}\tilde{X}^+)^n, \tag{3.16}$$

which is very similar to the universal $\mathcal{R}$ matrix of $U_q sl(2)$:

$$\mathcal{R}_{U_q sl(2)}=q^{-(1/2)H\otimes H}\sum_{n=0}^{\infty}\frac{(1-q^2)^n}{[n]_q!}q^{-(1/2)n(n-1)}(q^{(1/2)H}X^-)^n\otimes(q^{-(1/2)H}X^+)^n. \tag{3.17}$$

In fact, the similarity is not an accident but a consequence of QD. To see this, we first replace the generators $\tilde{X}^+$ and $\tilde{X}^-$ by e and f:[8]

$$e=q^{-H_2/2}\tilde{X}^+,\quad f=q^{H_1/2},\tilde{X}^-, \tag{3.18}$$

then Eqs. (3.1)–(3.4) become

$$[H_{1,2},e]=2e,\quad [H_{1,2},f]=-2f,$$

$$[e,f]=\frac{q^{H_1}-q^{-H_2}}{q^2-1}, \tag{3.19}$$

$$\Delta(H_{1,2})=H_{1,2}\otimes 1+1\otimes H_{1,2},\quad \Delta(1)=1\otimes 1,$$

$$\Delta(e)=e\otimes 1+q^{-H_2}\otimes e,\quad \Delta(f)=1\otimes f+f\otimes q^{H_1}. \tag{3.20}$$

$$S(H_{1,2})=-H_{1,2},\quad S(e)=-q^{H_2}e,\quad S(f)=-fq^{-H_1},\quad S(1)=1, \tag{3.21}$$

$$\varepsilon(H_{1,2})=\varepsilon(e)=\varepsilon(f)=0,\quad \varepsilon(1)=1. \tag{3.22}$$

These equations provide us the coefficients in the construction of a quantum double. Now, choosing the lower Borel subalgebra of $U_q gl(2)$,

$$U_q\mathcal{B}_-=\text{span}\{H_1^n f^m\}_{n,m=0}^{\infty}$$

as the Hopf algebra A in the quantum double construction, then by applying the same method as Tjin did in Ref. 8, we find that A^o can be identified with the upper Borel subalgebra

$$U_q\mathcal{B}_+=\text{span}\{H_2^n e^m\}_{n,m=0}^{\infty}.$$

This obtains the quantum double structure of $U_q gl(2)$.

Note that in the case of $U_q sl(2)$, the dual element of H can only be identified as an element proportional to H itself. However, in the $U_q gl(2)$ case, since J is a commuting element, it is possible to identify the dual element of H_1 as H_2, with

$$H_1-H_2\propto J,$$

and thus obtain the universal $\mathcal{R}$ matrix of Eq. (3.16).

IV. MULTIPARAMETER DEFORMATION AND RESHETIKHIN'S TWISTING TRANSFORMATION

The same multiparameter universal $\mathcal{R}$ matrix can also be obtained in a different way. In this section we shall discuss Reshetrikhin's twisting transformation,[35] and compare our definition of $U_q gl(2)$ with those introduced by other authors.[27,28,38,39] First denote $\mathcal{R}$ in (3.11) as $\mathcal{R}(H_1,H_2)$. Let $u=1$ and define $\tilde{H}=H-\alpha J$ with $q^{\tilde{H}}=q^H q^{-\alpha J}=q^H t^{-J}$. We obtain a single-parameter $U_q gl(2)$ and the corresponding universal $\mathcal{R}$ matrix now denoted by $\mathcal{R}(\tilde{H},\tilde{H})$:

$$\mathcal{R}(\tilde{H},\tilde{H})=q^{-(1/2)\tilde{H}\otimes\tilde{H}}\sum_{n=0}^{\infty}\frac{(1-q^2)^n}{[n]_q!}q^{-(1/2)n(n-1)}(q^{(1/2)\tilde{H}}\tilde{X}^-)^n\otimes(q^{-(1/2)\tilde{H}}\tilde{X}^+)^n. \tag{4.1}$$

According to the procedure introduced by Reshetikhin, for a QTHA A, if we can find an element $\mathcal{F}=\Sigma_i f^i\otimes f_i\in A\otimes A$ such that

$$\begin{gathered}(\Delta\otimes\mathrm{id})\mathcal{F}=\mathcal{F}_{13}\mathcal{F}_{23},\quad (\mathrm{id}\otimes\Delta)\mathcal{F}=\mathcal{F}_{13}\mathcal{F}_{12},\\ \mathcal{F}_{12}\mathcal{F}_{13}\mathcal{F}_{23}=\mathcal{F}_{23}\mathcal{F}_{13}\mathcal{F}_{12},\quad \mathcal{F}_{12}\mathcal{F}_{21}=1,\end{gathered} \tag{4.2}$$

then a possible multiparameter version of this QTHA can be established and the transformed universal $\mathcal{R}$ matrix can be obtained:

$$\mathcal{R}^{(\mathcal{F})}=\mathcal{F}^{-1}\mathcal{R}\mathcal{F}^{-1}, \tag{4.3}$$

where $\mathcal{R}$ represents the original single-parameter universal $\mathcal{R}$ matrix. For the present single-parameter $U_q gl(2)$ case the appropriate $\mathcal{F}$ is

$$\mathcal{F}=u^{-(1/4)(\tilde{H}\otimes J-J\otimes\tilde{H})}=u^{-(1/4)(H\otimes J-J\otimes H)}. \tag{4.4}$$

After applying Reshetikhin's twisting transformation on $\mathcal{R}(\tilde{H},\tilde{H})$, the transformed Hopf algebra is indeed (3.1)–(3.4) and the new universal $\mathcal{R}$ matrix $\mathcal{R}^{(\mathcal{F})}$ is that which appeared in (3.11), i.e.,

$$\mathcal{R}^{(\mathcal{F})}=\mathcal{R}(H_1,H_2).$$

Note that in the expression of $\mathcal{R}_0$ [cf. Eq. (3.6)], the exponent of the parameter u has an *antisymmetric* form, which can be obtained from Reshetikhin's transformation [note that in Eq. (4.2) the restriction $\mathcal{F}_{21}=\mathcal{F}^{-1}$ is required]. In contrast, the exponent of the parameter t has a symmetric form that comes from the third formula of (3.1) and cannot be obtained from a twisting transformation. It also seems not possible to obtain an expression containing the parameter v via a twisting transformation. We conclude that twisting transformation is the multiparameter generalization of the *coproduct* structure, whereas our construction is the multiparameter generalization for both *product* and *coproduct* structures.

Now we compare our $U_q gl(2)$ algebra with those introduced by other authors.[27,28,38,39] First consider the algebra introduced by Burdik and Hellinger.[27] Denote their coproduct, antipode and universal $\mathcal{R}$ matrix as Δ_{BH}, S_{BH} and $\mathcal{R}_{\mathrm{BH}}$, respectively. Then, the following substitutions,

$$\begin{gathered}\tilde{H}\to 2J_0,\quad X^{\pm}\to J_{\pm},\quad J\to 2Z,\\ v\to s,\quad u\to q,\\ \Delta\to\Delta_{\mathrm{BH}},\quad S\to S_{\mathrm{BH}},\quad \overline{\mathcal{R}}\to\mathcal{R}_{\mathrm{BH}},\end{gathered} \tag{4.5}$$

recover their algebra and universal $\mathcal{R}$ matrix. As a second example we consider the algebra introduced by Chakrabarti and Jagannathan.[28] The replacements

$$\tilde{H}\to 2\tilde{J}_0,\quad \tilde{X}^{\pm}\to\tilde{J}_{\pm},\quad J\to 2\tilde{Z},$$
$$q\to Q,\quad v\to 1,\quad u\to\lambda^{-1},\tag{4.6}$$
$$\Delta\to\Delta_{\rm CJ},\quad S\to S_{\rm CJ},\quad \overline{\mathcal{R}}\to\mathcal{R}_{\rm CJ}$$

recover their results. The main differences between their results and ours are: (i) in the first example u is set equal to q, which is not an arbitrary parameter; (ii) in the second example there is no v-like parameter and (iii) there is an extra arbitrary parameter t in the commutation relation (3.1).

In some sense our expression of $U_q gl(2)$ is a ''general gauge form,'' whereas other authors have considered ''gauge fixing form'' of the same algebra. The advantage of our expression is that when we consider the representation theory of $U_q gl(2)$, more general and interesting matrix solutions of the Yang–Baxter equation are obtained.

V. THE HIGHEST WEIGHT REPRESENTATIONS OF $U_q gl(2)$

For the representation theory, we only study the highest weight representations.[15,32,38] Let π be the map from $U_q gl(2)$ to an m-dimensional $(m\geqslant 2)$ representation:

$$\pi(J)=\lambda\mathbf{1},\quad \pi(H)=\mu\mathbf{1}+\sum_{i=1}^{m}(m-2i+1)e_{ii},$$
$$\pi(\tilde{X}^{+})=\sum_{i=1}^{m-1}a_i\,e_{i,i+1},\quad \pi(\tilde{X}^{-})=\sum_{i=1}^{m-1}b_i\,e_{i+1,i}.\tag{5.1}$$

Here e_{ij} represents the matrix basis $[(e_{ij})_{kl}=\delta_{ik}\delta_{jl}]$ and $\mathbf{1}$ denotes the unit matrix. Our strategy is to find a proper choice of parameters λ, μ, $\{a_i,b_i\}_{i=0}^{m}$ such that they will give us the appropriate highest weight representations of $U_q gl(2)$. Substituting these expressions into (3.1), we get

$$a_ib_i=[i]_q\left(\frac{q^{\mu}q^{m-i}t^{-\lambda}-q^{-\mu}q^{i-m}t^{\lambda}}{q-q^{-1}}\right),\quad i=1,2,\ldots,m-1.\tag{5.2}$$

Here we do not require b_i to have any prior relation to a_i. Equation (5.2) naturally comes from the commutation relation (3.1) of $U_q gl(2)$. Let $t^{\lambda}=q^{\tau}$. Equation (5.2) can now be rewritten as

$$a_ib_i=[i]_q[\mu-\tau+m-i]_q,\quad i=1,2,\ldots,m-1.$$

For $i=m-1$, comparing with another expression [also obtained from (3.1)],

$$a_{m-1}b_{m-1}=-[\mu-\tau+1-m]_q,$$

and, using the identities

$$[x]_q^2-[y]_q^2=[x-y]_q[x+y]_q,$$
$$[x]_q[y]_q=\left[\frac{x+y}{2}\right]_q^2-\left[\frac{x-y}{2}\right]_q^2,$$

we find

$$[\mu-\tau]_q[m]_q=0.\tag{5.3}$$

This result thus gives us two kinds of highest weight representation:

Type a: If $q^{2(\mu-\tau)}=1$ or $q^{2\mu}t^{-2\lambda}=1$, then q can be any complex number.

Type b: If μ, τ or $q^{2\mu}t^{-2\lambda}$ are arbitrary complex numbers, then m satisfies $[m]_q=0$ or $q^{2m}=1$. In other words, q must be restricted to the roots of unity.

Now let us consider two simple examples. First, the $m=2$ case:

$$\pi(H)=\begin{pmatrix}\mu+1 & 0\\ 0 & \mu-1\end{pmatrix},\quad \pi(J)=\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$
$$\pi(\tilde{X}^{+})=\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix},\quad \pi(\tilde{X}^{-})=\begin{pmatrix}0 & 0\\ b & 0\end{pmatrix}, \tag{5.4}$$

$$ab=\frac{q^{\mu+1}t^{-\lambda}-q^{-\mu-1}t^{\lambda}}{q-q^{-1}}. \tag{5.5}$$

The 4×4 matrix solutions R of the YBE can be obtained via the representation $R=(\pi\otimes\pi)\mathcal{R}$:

$$R=q^{-(1/2)(\mu^2-1)}t^{\lambda\mu}\begin{pmatrix}q^{-1}(q^{-\mu}t^{\lambda}) & 0 & 0 & 0\\ 0 & u^{\lambda} & 0 & 0\\ 0 & (q^{-1}-q)ab & u^{-\lambda} & 0\\ 0 & 0 & 0 & q^{-1}(q^{\mu}t^{-\lambda})\end{pmatrix}. \tag{5.6}$$

Let $q^{\mu}t^{-\lambda}=q^{-1}s$, $u^{\lambda}=\gamma$ and drop the factor $q^{-(1/2)(\mu^2-1)}t^{\lambda\mu}$. We then have

$$R=\begin{pmatrix}s^{-1} & 0 & 0 & 0\\ 0 & \gamma & 0 & 0\\ 0 & s^{-1}-s & \gamma^{-1} & 0\\ 0 & 0 & 0 & q^{-2}s\end{pmatrix}. \tag{5.7}$$

According to previous discussions, this R matrix in fact represents two solutions, which are

$$R_a=\begin{pmatrix}s^{-1} & 0 & 0 & 0\\ 0 & \gamma & 0 & 0\\ 0 & s^{-1}-s & \gamma^{-1} & 0\\ 0 & 0 & 0 & s^{-1}\end{pmatrix},\quad R_b=\begin{pmatrix}s^{-1} & 0 & 0 & 0\\ 0 & \gamma & 0 & 0\\ 0 & s^{-1}-s & \gamma^{-1} & 0\\ 0 & 0 & 0 & -s\end{pmatrix}. \tag{5.8}$$

When q is generic (type a), we have $q^{-2}s^2=1$, which gives us solution R_a. On the other hand, if s is arbitrary (type b), we have $q^4=1$, which implies $q^2=-1$ ($q^2=1$ is ruled out since that will cause $ab\to\infty$) and gives us solution R_b. Next, we consider the $m=3$ case,

$$\pi(H)=\begin{pmatrix}\mu+2 & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \mu-2\end{pmatrix},\quad \pi(J)=\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$
$$\pi(\tilde{X}^{+})=\begin{pmatrix}0 & a_1 & 0\\ 0 & 0 & a_2\\ 0 & 0 & 0\end{pmatrix},\quad \pi(\tilde{X}^{-})=\begin{pmatrix}0 & 0 & 0\\ b_1 & 0 & 0\\ 0 & b_2 & 0\end{pmatrix}, \tag{5.9}$$

$$a_1 b_1=[\mu-\tau+2]_q=\left(\frac{q^{\mu+2}t^{-\lambda}-q^{-\mu-2}t^{\lambda}}{q-q^{-1}}\right),\quad t^{\lambda}=q^{\tau},$$
$$a_2 b_2=[2]_q[\mu-\tau+1]_q=(q+q^{-1})\left(\frac{q^{\mu+1}t^{-\lambda}-q^{-\mu-1}t^{\lambda}}{q-q^{-1}}\right). \tag{5.10}$$

Let $q^{-\mu}t^{\lambda}=q^2 s^{-2}$, $u^{\lambda}=\gamma$ and remove the factor $q^{-(1/2)\mu^2}t^{\lambda\mu}$. Then we get

$$R=\begin{pmatrix} A_1 & 0 & 0\\ B_1 & A_2 & 0\\ C & B_2 & A_3\end{pmatrix}, \tag{5.11}$$

where A_1, A_2, A_3, B_1, B_2 and C are 3×3 matrices:

$$A_1=\begin{pmatrix} q^2 s^{-4} & 0 & 0\\ 0 & q^2 s^{-2}\gamma & 0\\ 0 & 0 & q^2\gamma^2\end{pmatrix},\quad A_2=\begin{pmatrix} q^2 s^{-2}\gamma^{-1} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & q^{-2}s^2\gamma\end{pmatrix},$$
$$A_3=\begin{pmatrix} q^2\gamma^{-2} & 0 & 0\\ 0 & q^{-2}s^2\gamma^{-1} & 0\\ 0 & 0 & q^{-6}s^4\end{pmatrix}, \tag{5.12}$$

$$B_1=\begin{pmatrix} 0 & q^2(s^{-4}-1) & 0\\ 0 & 0 & (1-q^2)\gamma a_2 b_1\\ 0 & 0 & 0\end{pmatrix},$$
$$B_2=\begin{pmatrix} 0 & (1-q^2)\gamma^{-1}a_1 b_2 & 0\\ 0 & 0 & (1+q^{-2})(1-q^{-2}s^4)\\ 0 & 0 & 0\end{pmatrix}, \tag{5.13}$$

$$C=\begin{pmatrix} 0 & 0 & (s^{-4}-1)(q^2-s^4)\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{5.14}$$

This result also provides us two kinds of R matrices. When $(q/s)^4=1$, we have the type a solution (the standard solution), whereas in the situation $(q/s)^4\neq 1$, we have $q^6=1\rightarrow 1+q^2+q^4=0$, which gives us type b solution (the nonstandard solution). Note that the factors a_1b_2 and a_2b_1 appearing in B_1 and B_2 cannot be uniquely determined in terms of q,γ,s only, whereas their product $(a_1b_2a_2b_1)=(a_1b_1a_2b_2)$ is unique. The type a solution is well known and can be obtained by different methods. The type b solutions are also known by many authors.[15–20,32–34] However, most of the authors obtain type b solutions via solving matrix equations and do not emphasize their algebraic origin. Some authors use the ‘‘q-boson realization’’ method combined with representation theory.[32,33] Nevertheless, this method destroys the Hopf algebra structure.

For a general integer m, after removing the factor $q^{-(1/2)\mu^2}t^{\lambda\mu}$, and letting

$$q^{\mu}t^{-\lambda}=(q^{-1}s)^{m-1},\quad u^{\lambda}=\gamma, \tag{5.15}$$

we have

$$R=q^{(1/2)(m-1)^2}s^{-(m^2-1)}\sum_{n=0}^{m-1}\frac{(1-q^2)^n}{\{n\}_{q^2}!}q^n\sum_{i,j=1}^{m-n}q^{-2(i-1)(j-1)-n(i+j)}$$
$$\times s^{(m-1)(i+j+n)}\gamma^{-(i-j)}(a_jb_i)\cdots(a_{j+n-1}b_{i+n-1})\,e_{i+n,i}\otimes e_{j,j+n}, \tag{5.16}$$

where

$$a_ib_i=[i]_q\left(\frac{q^{\mu}t^{-\lambda}q^{m-i}-q^{-\mu}t^{\lambda}q^{i-m}}{q-q^{-1}}\right)$$
$$=[i]_q\left(\frac{s^{m-1}q^{1-i}-s^{1-m}q^{i-1}}{q-q^{-1}}\right) \tag{5.17}$$

and the identity

$$(q^{2\mu}t^{-2\lambda}-1)[m]_q=\left(\left(\frac{s}{q}\right)^{2(m-1)}-1\right)[m]_q=0 \tag{5.18}$$

holds. Here we define $(a_jb_i)\cdots(a_{j+n-1}b_{i+n-1})\equiv 1$ when $n=0$.

In the last section we see that Reshetikhin's twisting transformation transforms the single-parameter expression of a Hopf algebra into its multiparameter deformation form. Hence a natural question may arise: when we consider the representation of a Hopf algebra, in what sense does the twisting transformation generalize the R matrix solution of the Yang–Baxter equation? In order to answer this question, we now study the representations of $\mathcal{F}$. Denote the representation of $\mathcal{F}$ as F (here $\mathcal{F}=u^{-(1/4)(H\otimes J-J\otimes H)}$), i.e., $F=(\pi\otimes\pi)\mathcal{F}$. For the m-dimensional representation we have

$$F=\gamma^{-(1/4)(\pi(H)\otimes 1-1\otimes\pi(H))}=\sum_{i,j=1}^{m}\gamma^{(1/2)(i-j)}e_{ii}\otimes e_{jj}, \tag{5.19}$$

where the relation $u^{\lambda}=\gamma$ is used and one finds that F contains only parameter γ. As an example let us consider the $m=2$ case. The F matrix now reads

$$F=\begin{pmatrix}1&0&0&0\\0&\gamma^{-1/2}&0&0\\0&0&\gamma^{1/2}&0\\0&0&0&1\end{pmatrix}. \tag{5.20}$$

The representation of the $\mathcal{R}(\tilde{H},\tilde{H})$ that mentioned in Sec. IV, denoted as $R_{\tilde{H}\tilde{H}}$, is

$$R_{\tilde{H}\tilde{H}}=c(t,\mu,\lambda)\begin{pmatrix}s^{-1}&0&0&0\\0&1&0&0\\0&s^{-1}-s&1&0\\0&0&0&q^{-2}s\end{pmatrix}. \tag{5.21}$$

Here $c(t,\mu,\lambda)$ is an unimportant factor and can be ignored. Now the transformed R matrix $R^{(F)}$,

$$R^{(F)}=F^{-1}R_{\tilde{H}\tilde{H}}F^{-1}=\begin{pmatrix}s^{-1}&0&0&0\\0&\gamma&0&0\\0&s^{-1}-s&\gamma^{-1}&0\\0&0&0&q^{-2}s\end{pmatrix}, \tag{5.22}$$

is indeed the R in (5.7).

From these observations we see that only the parameter γ can appear in the representations of $\mathcal{F}$. Thus we conclude that: (i) Reshetikhin's twisting transformation transforms an R solution without parameter γ to a new solution with γ; (ii) The twisting transformation cannot change the type of representation, i.e., it is impossible to obtain an R matrix of type b from an R matrix of type a via a twisting transformation.

One more point should be emphasized: there are two sets of free parameters, one for the algebra $U_q gl(2)$ itself, they are t, u and v, and the other for the representation of $U_q gl(2)$, they are s and γ. In this article we distinguish these two sets of parameters explicitly.

VI. COLORED SOLUTIONS OF THE YANG–BAXTER EQUATION

In order to obtain a colored solution of the YBE via representation, we have to prepare two representations of $U_q gl(2)$:[32,38] $\pi_1=\pi^{\mu,\lambda}$ and $\pi_2=\pi^{\mu',\lambda'}$ acting on the first and second factors of the direct product $\otimes$, respectively. Then the colored solution is given by

$$R(\mu,\lambda;\mu',\lambda')=(\pi_1\otimes\pi_2)\mathcal{R}. \tag{6.1}$$

Now let us calculate $R(\mu,\lambda;\mu',\lambda')$. For the first factor, we have

$$\pi_1(H)=\sum_{i=1}^{m}(\mu+m-2i+1)e_{ii},\quad \pi_1(J)=\lambda\mathbf{1}=\lambda\sum_{i=1}^{m}e_{ii},$$

$$\pi_1(\tilde{X}^-)=\sum_{i=1}^{m\text{-}1}b_i\,e_{i+1,i},$$

and, for the second factor, we have

$$\pi_2(H)=\sum_{i=1}^{m}(\mu'+m-2i+1)\,e_{ii},\quad \pi_2(J)=\lambda'\mathbf{1}=\lambda'\sum_{i=1}^{m}e_{ii},$$

$$\pi_2(\tilde{X}^+)=\sum_{i=1}^{m\text{-}1}a_i'\,e_{i,i+1}.$$

Here,

$$\begin{aligned}R(\mu,\lambda;\mu',\lambda')=f(\mu,\lambda;\mu',\lambda')&\sum_{n=0}^{m\text{-}1}\frac{(1-q^2)^n}{\{n\}_{q^2}!}q^n(ss')^{(n/2)(m-1)}\left(\frac{\gamma}{\gamma'}\right)^{n/2}\\&\times\sum_{i,j=1}^{m\text{-}n}q^{-2(i-1)(j-1)-n(i+j)}((s')^{m-1}(\gamma')^{-1})^i(s^{m-1}\gamma)^j\\&\times(a_j'b_i)\cdots(a_{j+n-1}'b_{i+n-1})\,e_{i+n,i}\otimes e_{j\cdot j+n},\end{aligned} \tag{6.2}$$

and s,s',γ,γ' are defined by

$$\left(\frac{s}{q}\right)^{m-1}=q^{\mu}t^{-\lambda},\quad \left(\frac{s'}{q}\right)^{m-1}=q^{\mu'}t^{-\lambda'},\quad \gamma=u^{\lambda},\quad \gamma'=u^{\lambda'}, \tag{6.3}$$

and the factor

$$f(\mu,\lambda;\mu',\lambda')=q^{-(1/2)\mu\mu'}t^{(1/2)(\mu\lambda'+\mu'\lambda)}u^{(1/2)(\mu\lambda'-\mu'\lambda)}\times q^{(1/2)(m-1)^2}(ss')^{-(1/2)(m^2-1)}\left(\frac{\gamma'}{\gamma}\right)^{(1/2)(m+1)} \tag{6.4}$$

is irrelevant and can be dropped.

As discussed in the previous section, there are two different types of solution: type a (q is generic) and type b (q is a root of unity). When $m=2$, let us compare our results with Hlavatý's solutions[40] (see also Ref. 25):

$$R_1(\lambda,\mu)=\phi(\lambda,\mu)\begin{pmatrix}1&0&0&0\\0&p^+(\lambda)&0&0\\0&(1-k)\xi(\lambda)/\xi(\mu)&k/p^+(\mu)&0\\0&0&0&p^+(\lambda)/p^+(\mu)\end{pmatrix}, \tag{6.5}$$

$$R_2(\lambda,\mu)=\phi(\lambda,\mu)\begin{pmatrix}1&0&0&0\\0&p^+(\lambda)&0&0\\0&W(\lambda,\mu)&p^-(\mu)&0\\0&0&0&-p^+(\lambda)p^-(\mu)\end{pmatrix}, \tag{6.6}$$

where

$$W(\lambda,\mu)=(1-p^+(\lambda)p^-(\lambda))\xi(\lambda)/\xi(\mu) \tag{6.7}$$

with $\xi(\lambda)$ being an arbitrary function.

(1) For type a,

$$R_a=q^2(\gamma/\gamma')\begin{pmatrix}1&0&0&0\\0&q\gamma&0&0\\0&\pm(1-q^2)(\gamma/\gamma')^{1/2}&q/\gamma'&0\\0&0&0&\gamma/\gamma'\end{pmatrix}, \tag{6.8}$$

which becomes R_1 when we define $p^+(\lambda)=q\gamma$, $p^+(\mu)=q\gamma'$, $k=q^2$, and $\xi(\lambda)/\xi(\mu)=\pm(\gamma/\gamma')^{1/2}$.

(2) For type b,

$$R_b=(ss')\left(\frac{\gamma}{\gamma'}\right)\begin{pmatrix}1&0&0&0\\0&s\gamma&0&0\\0&-2q(ss')^{1/2}(\gamma/\gamma')^{1/2}a'b&s'/\gamma'&0\\0&0&0&-ss'(\gamma/\gamma')\end{pmatrix}, \tag{6.9}$$

where $q^2=-1$, a', b are arbitrary C numbers. Let $p^+(\lambda)=s\gamma$, $p^+(\mu)=s'\gamma'$, $p^-(\lambda)=s/\gamma$, and $p^-(\mu)=s'/\gamma'$. We get the diagonal part of R_2. Furthermore, rewriting $a'b=a'ab/a$, using the relation $ab=(s-s^{-1})/(q-q^{-1})=(q/2s)(1-s^2)$, and defining

$$\frac{\xi(\lambda)}{\xi(\mu)}=\frac{[(\gamma/s)^{1/2}/a]}{[(\gamma'/s')^{1/2}/a']}, \tag{6.10}$$

we obtain $W(\lambda,\mu)=-2q(ss')^{1/2}(\gamma/\gamma')^{1/2}a'b$, which leads to the nonstandard solution R_2.

Another interesting application is to compare our solutions with those given in Ref. 32. Their universal $\mathcal{R}$ matrix (4.1) is our $\overline{\mathcal{R}}$. The equivalence can be easily established by the replacements:

$$2\hat{N}-\lambda_1\rightarrow H_1, \quad 2\hat{N}-\lambda_2\rightarrow H_2, \tag{6.11}$$

$$a^{\dagger}\cdot\alpha(\hat{N})\rightarrow\tilde{X}^{+}, \quad a\cdot\beta(\hat{N})\rightarrow\tilde{X}^{-}. \tag{6.12}$$

The additional relation

$$\alpha_i(\hat{N}-1)\cdot\beta_i(\hat{N})=[\lambda_i+1-\hat{N}]_q \tag{6.13}$$

appearing in Ref. 32 is a consistency condition, just like our Eqs. (5.17) and (5.18). Therefore, without explicit calculation, we know the solutions obtained in Ref. 32 are the same as (6.2).

When comparing the solution (6.2) with those in Refs. 18, 38, and 39, one has to be very careful. Since different authors sometimes adopt different definitions and conventions (for example, some authors define our RP or PR as their R, P which represents the permutation matrix), before the comparison one needs to choose appropriate conventions of $\{\Delta,S\}$ and definition of $\mathcal{R}$ or R.

VII. CONCLUDING REMARKS

We have studied the Hopf algebra structure and representation theory of a multiparameter version of $U_qgl(2)$. We show that the YBE can be solved directly in the QTHA framework, without introducing additional tricks or doing any transformations. The interesting feature of highest weight representation shows that there exist two kinds of representations. A large class of Borel type solutions R can be obtained via the highest weight representation, including standard and nonstandard colored solutions. We also study in what sense Reshetikhin's twisting transformation generalizes a single-parameter $\mathcal{R}$ (R) to a multiparameter $\mathcal{R}$ (R). However, in this article we have not yet discussed the cyclic representation[31,41,42] of $U_qgl(2)$ for q being a root of unity. We also have not explored what will happen to the $U_qgl(2)$ algebra itself and its universal $\mathcal{R}$ matrix when q is a root of unity.[43,44] We leave these discussions to another publication.

ACKNOWLEDGMENTS

We would like to thank M. H. Tu and D. H. Lin for reading the manuscript. This work is partly supported by Grant Nos. NSC 87-M-2112-008-002 and NSC88-M-2112-008-009 from the National Science Council (ROC).

After submitting the manuscript, we found the preprint math.QA/9911194 by D. Parashar and R. J. McDermott that partly overlaps with our work.

[1] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic, New York, 1982).
[2] Z. Q. Ma, *Yang-Baxter Equation and Quantum Enveloping Algebras*, Advanced Series on Theoretical Physical Science (World Scientific, Singapore, 1993), Vol. 1.
[3] V. G. Drinfeld, *Yang-Baxter Equation in Integrable Systems*, Advanced Series in Mathematical Physics, edited by M. Jimbo (World Scientific, Singapore, 1989), Vol. 10, p. 269.
[4] M. Jimbo, Lett. Math. Phys. **11**, 247 (1986).
[5] L. D. Faddeev, N. Yu. Reshetikhin, and L. A. Takhtajan, *Yang-Baxter Equation in Integrable Systems*, Advanced Series in Mathematical Physics, edited by M. Jimbo (World Scientific, Singapore, 1989), Vol. 10, p. 299.
[6] L. A. Takhtajan, ''Lectures on Quantum Groups'' in *Introduction to Quantum Group and Integrable Massive Models of Quantum Field Theory*, Nankai Lectures on Mathematical Physics, edited by M. L. Ge (World Scientific, Singapore, 1990).
[7] L. A. Takhtajan, *Introduction to Quantum Groups*, Lecture Notes in Physics (Springer-Verlag, New York, 1990), Vol. 370, p. 1.
[8] T. Tjin, Int. J. Mod. Phys. A **7**, 6175 (1992).
[9] J. Links, A. Foerster, and M. Karowski, J. Math. Phys. **40**, 726 (1999).
[10] P. P. Kulish and E. K. Sklyanin, J. Phys. A **24**, L435 (1991).
[11] L. Mezincescu and R. I. Nepomechie, J. Phys. A **24**, L17 (1991).
[12] V. Pasquier and H. Saleur, Nucl. Phys. B **330**, 523 (1990).
[13] L. H. Kauffman, *Knots and Physics* (World Scientific, Singapore, 1991).
[14] F. Y. Wu, Rev. Mod. Phys. **64**, 1099 (1992).

[15] H. C. Lee, *Invariants of quantum group $U_{qs}(gl(2;C))$ and Alexander–Conway link polynomial*, preprint, CRL TP-90-1123.
[16] Y. Akutsu and T. Deguchi, Phys. Rev. Lett. **67**, 777 (1991).
[17] H. C. Lee, J. Phys. A **29**, 393 (1996).
[18] M. L. Ge, G. C. Liu, and Y. W. Wang, J. Phys. A **26**, 4607 (1993).
[19] H. C. Lee, ''Hopf Algebra, Complexification of $U_q(sl(2,C))$ and Link Invariants,'' in *Fields, Strings and Quantum Gravity*, edited by H. Guo, Z. M. Qiu, and H. Tye (OPA, Amsterdam, 1990).
[20] M. Couture, H. C. Lee, and N. C. Schmeing, ''A New Family of N-State Representations of the Braid Group,'' in *Physics, Geometry and Topology*, edited by H. C. Lee (Pleum, New York, 1990), p. 573; M. Couture, M. L. Ge, and H. C. Lee, J. Phys. A **23**, 4765 (1990); M. Couture, M. L. Ge, H. C. Lee, and N. C. Schmeing, *ibid.* **23**, 4751 (1990).
[21] M. L. Ge, G. C. Liu, and K. Xue, J. Phys. A **24**, 2679 (1991).
[22] N. H. Jing, M. L. Ge, and Y. S. Wu, Lett. Math. Phys. **21**, 193 (1991).
[23] M. L. Ge and A. C. T. Wu, J. Phys. A **24**, L725 (1991).
[24] M. L. Ge and A. C. T. Wu, J. Phys. A **25**, L807 (1992).
[25] M. L. Ge and K. Xue, J. Phys. A **24**, L895 (1991).
[26] M. Couture, J. Phys. A **24**, L103 (1991).
[27] Č. Burdík and P. Hellinger, J. Phys. A **25**, L629 (1992).
[28] R. Chakrabarti and R. Jagannathan, J. Phys. A **27**, 2023 (1994).
[29] M. L. Ge, X. F. Liu, and C. P. Sun, Lett. Math. Phys. **23**, 169 (1991).
[30] M. L. Ge, X. F. Liu, and C. P. Sun, J. Math. Phys. **33**, 2541 (1992).
[31] M. L. Ge and H. C. Fu, J. Math. Phys. **33**, 427 (1992).
[32] M. L. Ge, C. P. Sun, and K. Xue, Int. J. Mod. Phys. A **7**, 6609 (1992).
[33] M. L. Ge and Y. W. Wang, J. Phys. A **26**, 443 (1993).
[34] N. H. Jing, Contemp. Math. **134**, 129 (1992).
[35] N. Reshetikhin, Lett. Math. Phys. **20**, 331 (1990).
[36] A. Schirrmacher, J. Phys. A **24**, L1249 (1991).
[37] A. Schirrmacher, J. Wess, and B. Zumino, Z. Phys. C **49**, 317 (1991).
[38] A. Kundu and B. Basu-Mallick, J. Phys. A **27**, 3091 (1994).
[39] Č. Burdík and P. Hellinger, J. Phys. A **25**, L1023 (1992).
[40] L. Hlavatý, *On the solution of the Yang-Baxter equations in Quantum Groups and related Topics*, Mathematical Physics Study, edited by Gielerak R. (Kluwer Academic, Dordrecht, 1992), Vol. 13, p. 179.
[41] M. Jimbo, *Topics from Representations of $U_q(g)$-An Introductory Guide to Physicists* in *Quantum Groups and Quantum Integrable Systems*, Nankai Lectures on Mathematical Physics, edited by M. L. Ge (World Scientific, Singapore, 1992).
[42] P. Roche and D. Arnaudon, Lett. Math. Phys. **17**, 295 (1989).
[43] C. Goméz and G. Sierra, Nucl. Phys. B **373**, 761 (1992).
[44] C. Goméz, M. Ruiz-Altaba, and G. Sierra, Phys. Lett. B **256**, 95 (1991).