LICKORISH INVARIANT AND QUANTUM OSP(1|2)

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Lickorish's method for constructing topological invariants of 3 - manifolds is generalized to the quantum supergroup setting. An invariant is obtained by applying this method to the Kauffman polynomial arising from the vector representation of $U_q(osp(1|2))$. A transparent proof is also given showing that this invariant is equivalent to the $U_q(osp(1|2))$ invariant obtained in an earlier publication.

Introduction

Since Jones' seminal work[1], the theories of knots and 3 - manifolds have made dramatical progress (See [2] for a review). By now several approachs are available for constructing the so called quantum invariants of 3 - manifolds, notably, the quantum field theoretical approach[3], the quantum group approach[4], Lickorish's recoupling theory[5], the $6j$ symbol method of Turaev - Viro[6], and the conformal field theoretical method[7]. All these approachs originated from theoretical physics, but they differ from one another very significantly in the mathematical formulations, and each method has its own advantages in addressing specific problems. Therefore, it is important to further develop the different approaches, even though it is believed that they all give rise to the same invariants of 3 - manifolds (The Turaev - Viro invariant is known to be the square of the norm of the Reshetikhin-Turaev invariant).

The recoupling theory was first introduced in [5] by Lickorish, who used the representation theory of the Temperley - Lieb algebra to reproduce the Jones - Witten - Reshetikhin - Turaev invariants. Since then the method has been developed extensively by other people[8][9][10]. In this letter, we aim to extend the recoupling theory in another direction, namely, to incorporate supersymmetry. We will also investigate the connection of the recoupling theory with the Reshetikhin - Turaev formalism in the quantum supergroup setting.

Parallel to the quantum group formulation of topological invariants of links and 3 - manifolds, there exists a supersymmetric version[11] based on the theory of Lie superalgebras and quantum supergroups. Due to the vast difference between the representation theory of quantum supergroups and that of the ordinary quantum groups, the associated topological invariants in both cases also exhibit different features. For example, there exist infinitely many families of multi - parameter generalizations of the Alexander - Conway invariant, arising from the so - called typical irreps. Such invariants can not be obtained within the framework of ordinary quantum groups. In this letter, we will apply the recoupling theory to the Kauffman polynomial associated with the vector representation of the quantum supergroup $U_q(osp(1|2))$ to construct the corresponding topological invariant of 3 - manifolds. We will also provide a transparent proof showing that the resultant invariant is equivalent to that constructed in [12]. In doing so, we establish a precise relationship between the recoupling theory and Reshetikhin - Turaev method within the context of our study.

Although we have limited ourselves to the quantum supergroup $U_q(osp(1|2))$ here, the method developed for constructing the Lickorish invariant can be readily applied to self - dual atypical irreps of any quantum supergroup. It also appears to be possible to extend the formalism to include typical irreps, which of course are much more interesting due to their connections with the generalized Alexander - Conway polynomials, and possibly generalized versions of the Casson invariant[13]. Results on this problem will be reported in a separate publication.

$U_q(osp(1|2))$

We will work on the complex field $\mathbf{C}$. Let q be an N - th primitive root of unity with N a positive odd integer satisfying

$$N = 2r + 1, \quad 0 < r \in \mathbf{Z}_+.$$

The quantum supergroup $U_q(osp(1|2))$ is a $\mathbf{Z}_2$ graded Hopf algebra, with the underlying algebra generated by $\{e, f, K^{\pm 1}\}$ subject to the relations

$$\begin{aligned} ef + fe &= \frac{K - K^{-1}}{q - q^{-1}}, \\ KeK^{-1} &= qe, \\ KfK^{-1} &= q^{-1}f, \\ e^{2N} = f^{2N} &= 0, \\ K^{\pm N} &= 1, \end{aligned} \tag{1}$$

where the elements e and f are odd, while $K^{\pm 1}$ are even. The co - multiplication $\Delta : U_q(osp(1|2)) \to U_q(osp(1|2)) \otimes U_q(osp(1|2))$ is given by

$$\begin{aligned} \Delta(e) &= e \otimes K + 1 \otimes e, \\ \Delta(f) &= f \otimes 1 + K^{-1} \otimes f, \\ \Delta(K^{\pm}) &= K^{\pm 1} \otimes K^{\pm 1}. \end{aligned}$$

It is well known that the $U_q(osp(1|2))$ so defined has the structures of a quasi triangular $\mathbf{Z}_2$ graded Hopf algebra. We denote by R its universal R matrix and set $R = \sum \alpha_t \otimes \beta_t$. Then R provides an isomorphism between the two algebras $\Delta[U_q(osp(1|2))]$ and $\Delta'[U_q(osp(1|2))]$, where Δ' is the opposite co - multiplication. Furthermore, R also satisfies the quantum Yang - Baxter equation. Define $v = K^{-1}\sum_t (-1)^{[\alpha_t]} S(\beta_t)\alpha_t$, where S is the antipode of $U_q(osp(1|2))$. Then v belongs to the center of $U_q(osp(1|2))$.

Any $U_q(osp(1|2))$ module W is a $\mathbf{Z}_2$ graded vector space $W = W_0 \oplus W_1$, where W_0 and W_1 are the even and odd subspaces respectively. Let W' be the $\mathbf{Z}_2$ graded vector space with $W'_0 = W_1$, and $W'_1 = W_0$. Then W' has a natural $U_q(osp(1|2))$ module structure. These two modules are evidently isomorphic, with the isomorphism given by a homogeneous degree 1 linear mapping. Let $f : W \to W$ be any module homomorphism. Then f gives rise to a module homomorphism $f : W' \to W'$ in a natural way. Now the q - superdimensions of f satisfy the following obvious relation

$$Str_W(Kf) \;=\; -Str_{W'}(Kf).$$

In [12], we classified all the irreducible representations of this quantum supergroup. It was shown that there exist only a finite number of irreducible representations, which are all finite dimensional, and each of them is uniquely characterized by an integer in $\mathbf{Z}_N = \{0, 1, ..., N-1\}$. Let $V^+(\lambda)$ be an irreducible $U_q(osp(1|2))$ module, then it possesses a unique highest weight vector $v_0(\lambda)$, which is assumed to be even, such that $ev_0(\lambda) = 0$, $Kv_0(\lambda) = q^\lambda v_0(\lambda)$, $\lambda \in \mathbf{Z}_N$. A basis for $V^+(\lambda)$ is given by $\{v_0(\lambda), v_1(\lambda), ..., v_{2\lambda}(\lambda)\}$, where $v_{i+1}(\lambda) = fv_i(\lambda)$, $fv_{2\lambda}(\lambda) = 0$. We will denote by $V^-(\lambda)$ the $U_q(osp(1|2))$ module isomorphic to $V^+(\lambda)$ but with an odd highest weight vector.

The q - superdimension of $V^+(\lambda)$ is given by

$$\begin{aligned} SD_q(\lambda) &:= Str_{V^+(\lambda)}(K) \\ &= \frac{q^{\lambda+1} + q^{-\lambda}}{q+1}. \end{aligned}$$

An important fact is that all the irreps have nonzero q - superdimensions, and for this reason, all the irreps had to be included in constructing the 3 - manifold invariant of [11]. However, the S - matrix arising from the Hopf link is singular, thus $U_q(osp(1|2))$ does not qualify as a $\mathbf{Z}_2$ graded modular Hopf algebra.

The smallest nontrivial irreducible $U_q(osp(1|2))$ module $V^+(1)$ will play an important role in the remainder of the note. We will denote it by V, and the associated irrep by π. It is assumed through out the paper that the highest weight vector of V is *even*. It is important to observe that the irrep is self dual, that is, there exists a homogeneous degree zero isomorphism between V and the dual module V^*. The tensor product module $V \otimes V$ decomposes into

$$V \otimes V \;=\; V^+(2) \oplus V^-(1) \oplus V^+(0).$$

Thus the braid generator $\check{R} = P(\pi \otimes \pi)R$ satisfies the following third order polynomial relation

$$(\check{R} - q)(\check{R} + q^{-1})(\check{R} - q^{-2}) \;=\; 0.$$

This in particular implies that the link invariant arises from V is the Kauffman polynomial.

Consider the central element of $U_q(osp(1|2))$ defined by

$$\hat{C}^{(k)} \quad = \quad Str_{V^{\otimes k}}[(\pi^{\otimes k}\Delta^{(k-1)} \otimes id)(v^{-1}K^{-1} \otimes 1)R^T R], \quad k \in \mathbf{Z}_+.$$

Acting on the irreducible $U_q(osp(1|2))$ module $V^+(\lambda)$, $\hat{C}^{(k)}$ takes the eigenvalue

$$\chi_\lambda(\hat{C}^{(k)}) \quad = \quad Str_{V^{\otimes k}}[\pi^{\otimes k}\Delta^{(k-1)}(v^{-1}K^{-2\lambda-1})].$$

Note that $\chi_\lambda(\hat{C}^{(k)})$ is a finite sum of powers of q, thus it is consistent to first evaluate $\chi_\lambda(\hat{C}^{(k)})$ at *generic* q then specialize it to the N - th root of unity. This way we obtain

$$\chi_\lambda(\hat{C}^{(k)}) \quad = \quad \sum_{j=0}^{k} b_j^{(k)} \frac{q^{(j+1)(2\lambda+1)} + q^{-j(2\lambda+1)}}{1 + q^{2\lambda+1}} q^{j(j+1)},$$

where $b_j^{(k)}$ are a set of complex numbers determined by the following recursion relations

$$\begin{aligned} b_j^{(n+1)} &= b_{j-1}^{(n)} - b_j^{(n)} + b_{j+1}^{(n)}, \quad j > 0 \\ b_0^{(n+1)} &= b_1^{(n)}, \end{aligned} \tag{2}$$

with the initial condition $b_0^{(0)} = 1$, $b_j^{(0)} = 0$, $\forall j > 0$. It is easy to see that

$$b_n^{(n)} = 1, \quad b_{n+j}^{(n)} = 0, \quad \forall j > 0.$$

Lickorish Invariant

We construct the Lickorish invariant of 3 - manifolds in this section. Although we only consider the quantum supergroup $U_q(osp(1|2))$, the method developed here is general, and can be readily applied to self - dual atypical irreps of any other quantum supergroups.

We will need some facts about the quantum supergroup approach to invariants of framed links, which we recall here. The Reshetikhin - Turaev approach to link invariants was generalized to quantum supergroups in [11], where a functor from the category of coloured ribbon graphs to the category of representations of $\mathbf{Z}_2$ graded ribbon Hopf algebras was constructed. In plain term, this functor associates each coloured ribbon graph with a homomorphism of $\mathbf{Z}_2$ graded modules of a quantum supergroup, where the modules are associated with the 'colour' of the graph. In particular, an over crossing is represented by the universal R matrix, and an under crossing by R^{-1}. The composition of coloured ribbon graphs corresponds to the composition of homomorphisms of quantum supergroup modules, and the juxtaposition of ribbon graphs to tensor product of module homomorphisms. The precise definition was given in [11] in explicit form, and we refer to that paper for details. Here we merely discuss a few aspects of the functor, which will be used extensively later.

Given any oriented (k, l) ribbon graph Γ, we colour each of its components by the vector module V of $U_q(osp(1|2))$. The Reshetikhin - Turaev functor maps the coloured ribbon graph to a module homomorphism $F(\Gamma_V) : V^{\otimes k} \to V^{\otimes l}$. Now reversing the orientation of any component of the ribbon graph will change the colouring module for that component to its dual module. As we have already pointed out, V is self - dual, thus $F(\Gamma_V)$ is independent of the orientation of Γ. Also, on any $(0,0)$ ribbon graph, this module homomorphism yields the Kauffman polynomial for ribbon graphs.

Consider the (k,k) ribbon graphs given in Figure 1 and Figure 2 respectively,

Figure 1 Figure 2

where Fig. 1 has n annuli. We colour the ribbons of Fig. 2 from left to right by the modules $W_1, ..., W_k$ respectively. We also colour the ribbons of Fig. 1 in the same way, but colour each of its annuli by V. Set $W = W_1 \otimes ... \otimes W_k$, and denote the resultant coloured coloured ribbon graphs of Fig. 1 and Fig. 2 by $\phi_W^{(n)}$ and ζ_W respectively. Then the functor F gives

$$\begin{aligned} F(\phi_W^{(n)}) &= \hat{C}^{(n)} : W \to W, \\ F(\zeta_W) &= v : W \to W, \end{aligned}$$

where the central elements $\hat{C}^{(n)}$ and v act on the tensor product module W via the co - multiplication $\Delta^{(k-1)}$.

Let us now construct the Lickorish invariant. Lickorish's construction uses two fundamental theorems from the theory of 3 - manifolds. In the earlier 1960s, Lickorish and Wallace proved that each framed link in S^3 determines a compact, closed, oriented 3 - manifold, and every such 3 - manifold is obtainable by surgery along a framed link in S^3[14]. Further advances along this line were obtained by Kirby, Craggs, and Fenn and Rourke[15][16], who proved that orientation preserving homeomorphism classes of compact, closed, oriented 3 - manifolds correspond bijectively to equivalence classes of framed links in S^3, where the equivalence relation is generated by the Kirby moves.

Let L be a framed link in S^3 with m components $L_1, ..., L_m$. We arbitrarily assign an orientation to each of its components. The resultant oriented framed link can be represented in a unique way by an oriented ribbon graph $\Gamma(L)$ [11]. Now we consider the oriented ribbon graph $\Gamma(L^{\{l_1,...,l_m\}})$ obtained from $\Gamma(L)$ in the following way: we replace each L_i of L by a cable of l_i copies of L_i with the same orientation, where $l_i \in \{0, 1, ..., N-1\}$. This leads to an oriented framed link with $\sum_{1\leq i\leq m} l_i$ components, the associated oriented ribbon graph of which is $\Gamma(L^{\{l_1,...,l_m\}})$. We colour each component of $\Gamma(L^{\{l_1,...,l_m\}})$ by the vector module V of $U_q(osp(1|2))$, and denote the corresponding coloured ribbon graph by $\Gamma(L^{\{l_1,...,l_m\}})_V$. Applying the Reshetikhin -

Turaev functor F to it leads to a complex number $F(\Gamma(L^{\{l_1,...,l_m\}})_V)$. The self duality of V implies that this number is independent of the arbitrarily chosen orientation of L.

Construct

$$\Sigma(L) \;=\; \sum_{l_1,...,l_m=0}^{N-1} \prod_{i=1}^{m} d^{(l_i)} \; F(\Gamma(L^{\{l_1,...,l_m\}})_V), \tag{3}$$

where the $d^{(l_i)}$ are a set of constants chosen in such a way that $\Sigma(L)$ is invariant under the positive Kirby moves. Needless to say, the critical problem is whether such $d^{(l_i)}$ exist. We will study this problem at great length later. Here let us take as granted their existence under the further assumption that

$$\begin{aligned} z \;&:=\; \sum_{l=0}^{N-1} d^{(l)} F(\Gamma(\mathcal{O}_{-1}^{\{l\}})_V) \\ &\neq\; 0, \end{aligned} \tag{4}$$

where $\mathcal{O}_{-1}$ represents the unknot with framing number -1, and $\mathcal{O}_{-1}^{\{l\}}$ the framed link obtained by extending the framed unknot to l parallel copies. Then

$$\mathcal{F}(M_L) \;=\; z^{-\sigma(A_L)}\Sigma(L), \tag{5}$$

is a topological invariant of the 3 - manifold M_L obtained by surgery along the framed link L. Here A_L is the linking matrix of L, and $\sigma(A_L)$ is the number of nonpositive eigenvalues of A_L.

Assuming the properties of $d^{(l)}$, we can easily show that $\mathcal{F}(M_L)$ is indeed a topological invariant of M_L: under the positive Kirby moves, $\sigma(A_L)$ is not changed, thus $\mathcal{F}(M_L)$ is invariant. Let L' denote the split link $L \cup \mathcal{O}_{-1}$. Then $\Sigma(L') = z\Sigma(L)$. Since $\sigma(A'_L) = \sigma(A_L)+1$, $\mathcal{F}(M_L)$ is invariant under the special negative Kirby move as well, and the proof is completed.

Let us now construct the d's. In Lickorish's original paper, the Jones polynomial was used for constructing the Witten - Reshetikhin - Turaev invariant. There the Jones - Wenzl theory of the Temperley - Lieb algebra played an important role in the determination of the constants $d^{(l)}$. In our case, the Lickorish invariant of 3 - manifolds is built from the Kauffman polynomial, which has a deep connection with the Birman - Wenzl algebra. It should be possible to obtain the $d^{(l)}$ by using only the representation theory of this algebra. However, such a method will not provide us with much information on the relationship between the Lickorish approach and the Reshetikhin - Turaev construction of 3 - manifold invariants. On the other hand, the representation theory of $U_q(osp(1|2))$ affords a common basis for both approaches, and also provides a much more powerful tool for determining the d's.

Consider the module homomorphisms associated with the coloured ribbon graphs of Figure 1 and Figure 2. Let $f : W \to W$ be any $U_q(osp(1|2))$ module homomorphism. Then the vanishing of the following q - supertrace

$$Str_W \left[Kf \left(\sum_{l=0}^{N-1} d^{(l)} F(\phi_W^{(l)}) - F(\zeta_W) \right) \right] \;=\; 0, \tag{6}$$

for all k, arbitrary W_i's and f, will guarantee the invariance of $\Sigma(L)$ under the positive Kirby moves. A sufficient condition for equation (6) to hold is that the central element of $U_q(osp(1|2))$ defined by

$$\delta \;=\; \sum_{l=0}^{N-1} d^{(l)}\hat{C}^{(l)} - v, \tag{7}$$

takes 0 eigenvalue in all irreducible representations of $U_q(osp(1|2))$, that is,

$$\sum_{l=0}^{N-1} d^{(l)}\chi_\lambda(\hat{C}^{(l)}) \;=\; q^{-\lambda(\lambda+1)}, \quad \forall\lambda\in \mathbf{Z}_N. \tag{8}$$

Therefore, the problem of constructing $\Sigma(L)$ is now reduced to that of solving equation (8). To do this, we introduce the $N\times N$ matrix $B=(b_{\mu\nu})_{\mu,\nu=0}^{N-1}$ with the entries given by $b_{\mu\nu}=b_\nu^{(\mu)}$, where $b_\nu^{(\mu)}$ are defined by the relations (2). Write $d=(d^{(0)},d^{(1)},...,d^{(N-1)})$, and define

$$b \;=\; dB.$$

Since B is lower triangular with all diagonal elements being 1, there exists a unique d corresponding to any given b. Now in terms of the components of b, equation (8) can be rewritten as

$$\sum_{\mu=0}^{N-1} b_\mu \frac{q^{(\mu+1)(2\lambda+1)}+q^{-\mu(2\lambda+1)}}{1+q^{2\lambda+1}} q^{\mu(\mu+1)} \;=\; q^{-\lambda(\lambda+1)}, \quad \lambda=0,1,...,N-1. \tag{9}$$

It is precisely this equation which appeared in the Reshetikhin - Turaev construction of 3 - manifold invariants using the quantum supergroup $U_q(osp(1|2))$. The most general solution of this equation was obtained in [12], which we quote below

$$b_\mu \;=\; \frac{(1+q)q^{\left(\frac{N+1}{2}\right)^2} G_{-1} SD_q(\mu)}{2N} + x_\mu - x_{N-\mu-1}, \quad \mu=0,1,...,N-1, \tag{10}$$

where x_μ are arbitrary complex parameters, $G_{-1}=\sum_{\lambda\in\mathbf{Z}_N} q^{-\lambda^2}$, and $SD_q(\mu)$ denotes the q - superdimension of $V^+(\mu)$.

To compute z, we apply the formula

$$\begin{aligned} F(\Gamma(\mathcal{O}_{-1}^{\{l\}})_V) &= Str_{V^{\otimes l}}(vK) \\ &= \sum_{j=0}^{l} b_j^{(l)} \frac{q^{j+1}+q^{-j}}{1+q} q^{-j(j+1)}, \end{aligned}$$

to cast it into the form

$$z \;=\; \sum_{\lambda=0}^{N-1} b_\lambda\, q^{-\lambda(\lambda+1)}\, SD_q(\lambda), \tag{11}$$

which coincides with the quantity also denoted by z in [12]. Now $z=q^{\frac{N+3}{2}}\left(\frac{G_{-1}}{\sqrt{N}}\right)^2$, which clearly has norm 1.

Lickorish invariant versus Reshetikhin - Turaev invariant

In the process of constructing the Lickorish invariant $\mathcal{F}(M_L)$, we have already noticed many similarities between this invariant and that of [12] obtained following a modified Reshetikhin - Turaev approach. Now we prove that these two invariants are actually equivalent, namely, on any compact closed orientable 3 - manifold, both invariants take the same value.

Consider the oriented ribbon graph corresponding to an oriented framed link L with m components L_i, $i = 1, 2, ..., m$. We colour the annulus associated with L_i by the $U_q(osp(1|2))$ module W_i, for all $i = 1, 2, ..., m$, and denote the resultant coloured ribbon graph by $\Gamma(L, \{W_1, ..., W_m\})$. Particularly interesting is the case when $W_i = V^{\otimes l_i}$, where V is the vector module of $U_q(osp(1|2))$, and $0 \leq l_i \leq N-1$. ($V^{\otimes 0} = \mathbf{C}$). As in the last section, we still denote by $\Gamma(L^{\{l_1,...,l_m\}})_V$ the coloured ribbon graph, which arises from the riented framed link $L^{\{l_1,...,l_m\}}$ obtained by extending each component L_i of L to a cable of l_i strands with the same orientation, and colouring the annulus associated with each strand by the vector module V of $U_q(osp(1|2))$. Then a moment's thinking reveals that

$$F(\Gamma(L^{\{l_1,...,l_m\}})_V) \;=\; F(\Gamma(L, \{V^{\otimes l_1}, ..., V^{\otimes l_m}\})).$$

To examine $\Gamma(L, \{V^{\otimes l_1}, ..., V^{\otimes l_m}\})$ more closely, we cut open one of its components, say, that associated with L_1, to obtain another coloured ribbon graph which we denote by $\Gamma(\check{L}, \{V^{\otimes l_1}, ..., V^{\otimes l_m}\})$. Then the module homomorphism $F(\Gamma(\check{L}, \{V^{\otimes l_1}, ..., V^{\otimes l_m}\}))$: $V^{\otimes l_1} \to V^{\otimes l_1}$ satisfies

$$F(\Gamma(L, \{V^{\otimes l_1}, ..., V^{\otimes l_m}\})) \;=\; Str_{V^{\otimes l_1}}\left[KF(\Gamma(\check{L}, \{V^{\otimes l_1}, ..., V^{\otimes l_m}\}))\right].$$

The right hand side can be evaluated by first decomposing the tensor product module $V^{\otimes l_1}$ into a direct sum of indecomposable $U_q(osp(1|2))$ modules, then taking the q - supertrace on each module separately. Since $l_1 \leq N-1$, $V^{\otimes l_1}$ is in fact completely reducible. Taking into account the parity of each irreducible submodule (i.e., the evenness or oddness of the highest weight vector), we have

$$\begin{aligned} & Str_{V^{\otimes l_1}}\left[KF(\Gamma(\check{L}, \{V^{\otimes l_1}, ..., V^{\otimes l_m}\}))\right] \\ =\; & \sum_{\mu=0}^{N-1} b_\mu^{(l_1)} Str_{V^+(\mu)}\left[KF(\Gamma(\check{L}, \{V^+(\mu), V^{\otimes l_2}, ..., V^{\otimes l_m}\}))\right] \\ =\; & \sum_{\mu=0}^{N-1} b_\mu^{(l_1)} F(\Gamma(L, \{V^+(\mu), V^{\otimes l_2}, ..., V^{\otimes l_m}\})), \end{aligned}$$

where $F(\Gamma(L, \{V^+(\mu), V^{\otimes l_2}, ..., V^{\otimes l_m}\}))$ represents the coloured ribbon graph arising from the oriented framed link L with the first component coloured by $V^+(\mu)$, and the rest by $V^{\otimes l_2}$, ..., $V^{\otimes l_m}$ respectively. It can be further expanded by decomposing $V^{\otimes l_2}$ etc. into direct sums of indecomposables. When $0 \leq l_i \leq N-1$, for all $i = 1, 2, ..., m$, we obtain the following important relation

$$F(\Gamma(L, \{V^{\otimes l_1}, ..., V^{\otimes l_m}\})) = \sum_{\mu_1,...,\mu_m=0}^{N-1} \prod_{i=1}^{m} b_{\mu_i}^{(l_i)}\; F(\Gamma(L, \{V^+(\mu_1), ..., V^+(\mu_m)\})). \quad (12)$$

Let us apply (12) to the definition (3) of $\Sigma(L)$. Recalling that $b = dB$, we immediately arrive at

$$\Sigma(L) \quad = \quad \sum_{\mu_1,...,\mu_m=0}^{N-1} \prod_{i=1}^{m} b_{\mu_i} \; F(\Gamma(L, \{V^+(\mu_1), ..., V^+(\mu_m)\})),$$

where the right hand side is precisely what appeared in [12]. Since the z defined by equation (4) coincides with the corresponding quantity there as well, we easily see that $\mathcal{F}(M_L)$ and the invariant of [12] are indeed the same.

References

[1] V. F. R. Jones, Bull. Amer. Math. Soc. **12** (1985) 103.

[2] V. G. Turaev, *Quantum Invariants of Knots and* 3 *- Manifolds*, Walter de Gruyter, Berlin, New York(1994).

[3] E. Witten, Commun. Math. Phys. **121** (1989) 351.

[4] N. Yu Reshetikhin and V. G. Turaev, Invent. Math. **103** (1991) 547.

[5] W. B. R. Lickorish, Pacific J. Math. **149** (1991) 337; Math. Ann. **290** (1991) 657; *Calculations with the Temperley - Lieb algebra*(preprint).

[6] V. G. Turaev and O. Y. Viro, Topology **31** (1992) 865.

[7] L. Crane, Commun. Math. Phys. **135** (1991) 615;
T. Kohno, Topology, **31** (1992) 203;
M. Kontsevich, *Rational conformal field theory and invariants of* 3 *- dimensional manifolds*, preprint.

[8] H. Wenzl, Invent. Math. **114** (1993) 235.

[9] C. Blanchet, N. Habegger, G. Masbaum and P. Vogel, Topology, **31** (1992) 685.

[10] L. Kauffman and S. L. Lens, *Temperley - Lieb recoupling theory and invariants of* 3 *- manifolds*, Princeton University Press, Princeton (1994).

[11] R. B. Zhang, Rev. Math. Phys., **7** (1995) 809.

[12] R. B. Zhang, Mod. Phys. Lett. **9** (1994) 1453.

[13] S. Akbulut and J. D. McCarthy, *Casson's invariant for homological 3- spheres: an exposition*, Mathematical Notes **36**, Princeton Univ. Press, Princeton (1990).

[14] W. B. R. Lickorish, Annals of Math. **76** (1962) 531.

[15] R. Kirby, Invent. Math. **45** (1978) 35.

[16] R. Fenn and C. Rourke, Topology **18** (1979) 1.