Anomalous Ultraviolet Divergences and Renormalizability of the Light-Cone Gauge

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The one-loop renormalizability of both the two-component (LC2) and four-component (LC4) formulations of the light-cone gauge is demonstrated by construction of the complete one-loop counter Lagrangeans. The Mandelstam-Leibbrandt prescription is used to regularize the singular $1/p^+$ factor. In LC4, the one-loop self-energy and three-vertex both have anomalous, unrenormalizable ultraviolet divergences, but the counterterms associated with these divergences cancel exactly, rendering the total counter Lagrangeans for the two formulations identical, at least to $O(g^3)$.

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The light-cone gauge[1] is a special axial gauge defined in terms of a null vector n_+

$$\mathbf{A}\cdot n_+ = \mathbf{A}^+ = 0, \quad n_+^2 = 0. \tag{1}$$

Although known for some time,[2] this gauge has acquired a certain notoriety for having Feynman integrals that are more singular than usual and therefore difficult to evaluate. This has changed following the recent discovery of a prescription[3,4] with which the integrals can be well regulated. Simultaneously, the gauge has become popular in the study of supersymmetric theories,[3,5-8] after Mandelstam[3] and Brink, Lindgren, and Nilsson[6] used it to prove the ultraviolet (uv) finiteness of the $N=4$ model. Recent interest in the light-cone gauge is further heightened by the recognition that it is the only gauge in which a quantum formulation of superstring theories is known.[9] However, the renormalizability of the simple Yang-Mills theory in the light-cone gauge has not yet been demonstrated. Indeed, recent calculations[7,8] have yield seemingly contradictory results, showing that, depending on which one of the two formulations (see below) one works with, the one-loop self-energy may or may not contain ultraviolet divergences that are unrenormalizable. The phenomenon of anomalous divergences is not well known, and its occurrence in the light-cone gauge has cast doubt on whether the gauge is really understood. The purpose of this paper is to study these anomalous divergences by computing the complete counter Lagrangean for the light-cone gauge at the one-loop level. It is shown that associated with the two- and three-point Green's functions there are anomalous, ultraviolet-divergent terms in one of the formulations. However, the total contribution of such terms to the counter Lagrangean cancels exactly to make the theory renormalizable and independent of the formulation.

A unique feature of the light-cone gauge is that, depending on the way the gauge constraint (1) is implemented, two distinct formulations are possible. The first formulation, LC4, results when (1) is implemented via a gauge-fixing Lagrangean, the practice for most other gauges. In this case the gauge field retains all four of its components even though the component $\mathbf{A}^+$ formally vanishes. The Feynman rules are the usual ones, except for the propagator,

$$\Delta_{\mu\nu}^{(0)ab}(p) = \frac{i\delta^{ab}}{p^2}[g_{\mu\nu} - (p_\mu n_{+\nu} + p_\nu n_{+\mu})/p^+], \tag{2}$$

which is especially singular because of the factor $1/p^+$.

The effective Lagrangean for the second formulation, LC2, is obtained by eliminating from the original Lagrangean both of the light-cone components $\mathbf{A}^+$ and $\mathbf{A}^-$ using (1) and the equation of motion

$$\partial^+\mathbf{A}^- = \partial^i\mathbf{A}^i + g(\partial^+)^{-1}(\mathbf{A}^i\times\partial^+\mathbf{A}^i) = \partial^i\mathbf{A}^i + g\mathbf{B}, \tag{3}$$

yielding (the bilinear $\mathbf{B}$ will appear repeatedly)[10]

$$\mathscr{L}_{\mathrm{LC2}} = -\tfrac{1}{2}(\mathbf{A}^i\circ\partial^2\mathbf{A}^i) - g[(\partial^j\mathbf{A}^i)\circ(\mathbf{A}^j\times\mathbf{A}^i) + (\partial^i\mathbf{A}^i)\circ\mathbf{B}] - \tfrac{1}{2}g^2\mathbf{B}^2 - \tfrac{1}{4}g^2(\mathbf{A}^i\times\mathbf{A}^j)^2. \tag{4}$$

The elimination of $\mathbf{A}^-$ from the Lagrangean is valid because it can be shown that (4) and the original full Lagrangean give rise to the same Hamiltonian.[10] It is vital to realize that, because (3) is inhomogeneous in the gauge field and in the coupling constant g, a term of a given order in A and g from the original (LC4) Lagrangean does not transform into terms of the same order in (4). The only term with a one-to-one mapping between the two formulations is the last term in (4), a fact that will be utilized later. In LC2 the propagator is very simple,

$$\Delta_{ij}^{(0)ab}(p) = -\frac{i}{p^2}\delta^{ab}\delta_{ij}, \tag{5}$$

but the three- and four-vertices are unusual,[6,7]

$$\Gamma^{(0)abc}_{ijk}(p,q,r)=gf^{abc}\{\delta_{ij}[(p-q)_k-r_k(p-q)^+/r^+]+\text{two cyclic terms}\},\tag{6}$$

$$\Gamma^{(0)abcd}_{ijkl}(p,q,r,s)=ig^2\left\{f^{abe}f^{cde}\left[\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}+\delta_{ij}\delta_{kl}\frac{(p-q)^+(r-s)^+}{(p+q)^+(r+s)^+}\right]+\text{two symmetric terms}\right\},\tag{7}$$

and also very singular for the same reason as (2).

The tensor algebras of the two formulations are very different; that of LC2 is two dimensional and that of LC4, in addition to being four dimensional, is complicated considerably by the presence of the two light-cone vectors n_+ and n_-.

The problem of finding a viable prescription to regularize the singular factor $1/p^+$ was solved by Mandelstam[3] and by Leibbrandt,[4] who independently devised prescriptions that can be shown to be equivalent.[11] In particular, Leibbrandt's prescription

$$1/p^+\to\lim_{\eta\to 0^+}[p^-/(p^+p^-+i\eta)]\tag{8}$$

is analogous to the prescription used to regularize the propagator for massive Dirac fermions. A key property of the Mandelstam-Leibbrandt (M-L) prescription is that it obeys the rule of power counting, a property that was used by Mandelstam[3] and by Brink, Lindgren, and Nilsson[6] to prove the finiteness of the $N=4$ supersymmetric model. It is worth noting that not all prescriptions are acceptable. For example, the principal-value prescription that works well for normal axial gauges[12,13] does not work for the light-cone gauge.[7,8,14]

Once a prescription is found, previously singular quantities in (2), (3), (6), and (7) become well behaved rendering all computations involving them, in particular the evaluation of Feynman integrals, straightforward in principle. The question, seriously raised by the discovery[8,10] of anomalous ultraviolet divergence in the self-energy in LC4, is whether the M-L prescription really works. One way to check this is to see if, for quantities that are gauge independent, the prescription yields the correct results in both LC2 and LC4. Here, we report the results of such a test where the complete one-loop counter Lagrangeans in both formulations are constructed by computing the one-loop, two-, three-, and four-point vertices. We also test the Slavnov-Taylor identities, but only in LC4 because in LC2, with the application of (1) and (3), all gauge degrees of freedom are removed and no such identities can be derived.

The calculation is simplified tremendously by the observation that all Feynman integrals needed can be reduced to a generalized integral with a known analytic representation:

$$\int d^{2\omega}q\,[(p-q)^2]^\kappa(q^2)^\mu(q^+)^\nu(q^-)^\lambda=\frac{i(\pi e^{-i\pi})^\omega(p^2)^{\omega+\kappa+\mu}(p^+)^\nu(p^-)^\lambda\Gamma(1+\lambda)}{\Gamma(-\kappa)\Gamma(-\mu)\Gamma(-\nu)\Gamma(2\omega+\kappa+\mu+\nu+\lambda)}$$
$$\times z^{-\nu}G^{2,3}_{3,3}\left(z\,\middle|^{1+\nu,\,1-\omega-\mu-\lambda,\,1+\omega+\kappa+\mu+\nu;}_{0,\,\omega+\kappa+\nu;\,\nu-\lambda}\right),\qquad z=2p^+p^-/p^2,\tag{9}$$

where the exponents κ, μ, ν, and λ as well as the generalized dimension ω are continuous variables and G is a Meijer function with well-known properties. For a discussion of the method of "analytic regularization" using the G-function representation see Lee and Milgram.[13–15] Details of the derivation of (9) and of the calculation leading to the results that follow will be reported elsewhere.[11]

For LC2 we obtain the infinite parts ($\epsilon=\omega-2$; for complete expressions for the vertex functions including the finite parts see Ref. 11)

$$\Pi^{(1)ab}_{ij}(p)|_{\text{inf}}=i\tfrac{11}{3}g^2Z_\epsilon\delta^{ab}\delta_{ij}p^2,\quad Z_\epsilon=C_2/(16\pi^2\epsilon);\tag{10}$$

$$\Gamma^{(1)abc}_{ijk}(p,-p,0)|_{\text{inf}}=-\tfrac{22}{3}g^3Z_\epsilon f^{abc}\delta_{ij}p_k;\tag{11}$$

$$\Gamma^{(1)abcd}_{ijkl}(p,-p,0,0)|_{\text{inf}}=-ig^4Z_\epsilon\{[\tfrac{22}{3}+(4)_{\text{ir}}]f^{abe}f^{cde}(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{kj})+\tfrac{11}{3}(f^{ace}f^{bde}-f^{ade}f^{cbe})\delta_{ij}\delta_{kl}\};\tag{12}$$

which, except for the infrared-infinite term $(4)_{\text{ir}}$, can all be canceled by corresponding terms generated from the counter Lagrangean

$$\delta\mathcal{L}_{\text{LC2}}=\delta\mathcal{L}^{(2+3+4)}_{\text{LC2}}=\tfrac{11}{3}g^2Z_\epsilon\mathcal{L}_{\text{LC2}}\tag{13}$$

with a single renormalization constant identical to that for the normal axial gauge.[13] This establishes the renormalizability of LC2 at the one-loop level. The results (11) and (12), which are new [we mention in passing that the uv and ir divergences in (12) are separated analytically[13] with use of (9)], also confirm that the transformation (3) and the Feynman rules (6) and (7) in conjunction with the prescription (8) give a consistent theory; so far we have encountered no surprises.

For LC4 we find $(r_\mu = p^- n_{+\mu},\ s_\mu = p^+ n_{-\mu})$

$$\Pi^{(1)ab}_{\mu\nu}(p)\big|_{\rm inf} = ig^2 Z_\epsilon \delta^{ab}\{-\tfrac{11}{3}(p^2 g_{\mu\nu} - p_\mu p_\nu) + 2[p_\mu (r-s)_\nu + p_\nu (r-s)_\mu] - 8r_\mu r_\nu/z + 4(r_\mu s_\nu + r_\nu s_\mu)/z\}, \tag{14}$$

$$\Gamma^{(1)ab}_{\lambda\mu\nu}(p,-p,0)\big|_{\rm inf} = g^3 Z_\epsilon f^{abc}\{-\tfrac{11}{3}(2g_{\lambda\mu}p_\nu - g_{\mu\nu}p_\lambda - g_{\nu\lambda}p_\mu) + 4g_{\lambda\mu}s_\nu + 2[g_{\mu\nu}(r-s)_\lambda + g_{\nu\lambda}(r-s)_\mu] + [-16r_\lambda r_\mu p_\nu + 8(r_\lambda s_\mu + r_\mu s_\lambda)p_\nu - 4(s_\lambda p_\mu + s_\mu p_\lambda)r_\nu - 4(p_\lambda r_\mu + p_\mu r_\lambda)s_\nu]/(zp^2) + 16 r_\lambda r_\mu r_\nu/(z^2 p^2)\}. \tag{15}$$

The infinite part of the four-vertex was not computed directly but will be deduced later. Because of (1), the n_+ (or r) dependent terms in (14) and (15) do not contribute to the counterterm, constructed essentially by contracting the vertices with the appropriate number of gauge fields. On the other hand, their presence is crucial to the Slavnov-Taylor identities[16]

$$p_\mu \Pi^{(1)ab}_{\mu\nu}(p) = 0, \quad p_\lambda \Gamma^{(1)}_{\lambda\mu\nu}abc(p,-p,0) = igf^{abc}\Pi^{(1)dc}_{\mu\nu}(p), \tag{16}$$

both of which are satisfied for (14) and (15). The n_- (or s) dependent terms, which do contribute to the counterterm, are anomalous since such terms are absent from the bare vertices. Indeed, in partial counter Lagrangean derived from (14) and (15),

$$\delta\mathscr{L}^{(2+3)}_{\rm LC4} = \tfrac{11}{3}g^2 Z_\epsilon\{\tfrac{1}{2}[(\mathbf{A}_\mu \circ \partial^2 \mathbf{A}^\mu) + (\partial_\mu \mathbf{A}^\mu)^2] - g(\partial_\mu \mathbf{A}_\nu)\circ(\mathbf{A}^\mu \times \mathbf{A}^\nu)\} - 2g^2 Z_\epsilon (\partial_\mu \mathbf{A}^\mu)\circ(\partial^+ \mathbf{A}^-) + 2g^3 Z_\epsilon (\partial^+ \mathbf{A}_\mu)\circ(\mathbf{A}^- \times \mathbf{A}^\mu), \tag{17}$$

the last two terms are easily recognized as being anomalous.

How do we reconcile the existence of these terms in LC4 with the fact that LC2, and therefore the light-cone gauge, is renormalizable? The answer is simple: Although the two anomalous terms look different, because of (3) they are actually identical in LC2. Thus

$$(\partial_\mu \mathbf{A}^\mu)\circ(\partial^+ \mathbf{A}^-) = g(\partial^+ \mathbf{A}_\mu)\circ(\mathbf{A}^- \times \mathbf{A}^\mu) = g(\partial^i \mathbf{A}^i)\circ \mathbf{B} + g^2 \mathbf{B}^2, \tag{18}$$

and the two terms cancel exactly from (17) which then has the form that assures the renormalizability of LC4 to $O(g^3)$. This further strengthens our belief that (3) can be used at any time to transform LC4 into LC2. The inverse transformation is not generally possible; the one used in the following is an exception. If we accept the equivalence of LC2 and LC4, then the counterterm corresponding to the four-vertex in LC4 can be computed by subtraction of (17) from (13) with the aid of (3) to obtain

$$\delta\mathscr{L}^{(4)}_{\rm LC4} = \delta L_{\rm LC2} - \delta\mathscr{L}^{(2+3)}_{\rm LC4} = -\tfrac{11}{3}g^4 Z_\epsilon \times \tfrac{1}{4}(\mathbf{A}^i \times \mathbf{A}^j)^2 = -\tfrac{11}{3}g^4 Z_\epsilon \times \tfrac{1}{4}(\mathbf{A}_\mu \times \mathbf{A}_\nu)^2. \tag{19}$$

The last equality is based on the fact that for the two terms in question the mapping between the two formulations is one-to-one. The last expression in (19), being exactly proportional to the $O(g^2A^4)$ term in the original $\mathscr{L}_{\rm LC4}$, allows one to write, without further ado,

$$\Gamma^{(1)abcd}_{\lambda\kappa\mu\nu}(p,q,r) = -\tfrac{11}{3}g^2 Z_\epsilon \Gamma^{(0)abcd}_{\lambda\kappa\mu\nu}(p,q,r). \tag{20}$$

The only thing that remains to be done to complete the demonstration that LC4 is also one-loop renormalizable is the verification of (20) by direct computation.

We close with a few remarks. (i) Slavnov-Taylor identities provide constraints on vertex functions but are not sufficiently restrictive to determine even the infinite parts of such functions. For example, given (14), the second identity in (16) does not uniquely determine the infinite part of the three-vertex (15). (ii) The possession of anomalous divergence seems to be a unique feature of LC4. It is clear that only theories with redundant degrees of freedom may have anomalous divergences. Thus LC2 cannot have such divergences. On the other hand, the normal axial gauge might have such divergences, but does not.[13] (iii) Because canceling anomalous divergences are rarely encountered, it is not always remembered that renormalizability is determined by the structure of the counter Lagrangean which should be gauge invariant, and not by the infinite parts of individual Green's functions which are not gauge invariant. The light-cone gauge, in particular LC4, illustrates clearly that renormalization is a process that is operative order-by-order in the number of loops, but not necessarily in powers of the coupling constant. (iv) In spite of the unfamiliar Feynman rules, the computation in LC2 is exceedingly simple. With the technical difficulty associated with the factor $1/p^+$ overcome, LC2 is probably the simplest of all gauges in which to work. In comparison, computations in LC4 are invariably much lengthier.

Note added.—The infinite parts of the general three-vertex $\Gamma^{(1)abc}_{\mu\nu\lambda}(p,q,r)$ in LC4 for arbitrary momenta satisfying $p+q+r=0$ has recently been computed by Dalbosco[17] and by Lee *et al.*,[18] yielding results that are in agreement. The three-vertex satisfies the generalized version of the three-point Slavnov-Taylor identity (16) and the A^+-independent counter Lagrangean generated by it is identical to that given in (17).

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[1]We work in the light-cone coordinates $n_\pm=(1,0,0,\pm1)/\sqrt{2}$; for any vector p, $p^\pm \equiv p\cdot n_\pm$; Lorentz indices μ, ν, etc. run from 0 to 3; indices i,j, etc. run from 1 to 2; **A** denotes a vector in the gauge group; $\mathbf{A}\circ\mathbf{B}\equiv A^aB^a$; $(\mathbf{A}\times\mathbf{B})^a=f^{abc}A^bB^c$.

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