Spontaneous Breaking of Topological Symmetry

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The spontaneous breaking of topological symmetry induced by a vanishingly small symmetry breaking term is investigated. It is shown that, in the presence of Gribov zero modes, topological theories without smearing terms, which are inequivalent to theories with smearing terms, permit spontaneous symmetry breaking. The relation with reducible configurations is briefly discussed.

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One of the motivations for developing topological field theories [1] is the hope that these models can describe the highly symmetrical phases of some realistic, less symmetric, field theories. The main obstacle to this approach is the difficulty in finding a way to break topological symmetry [1]. Hitherto, the models that have been investigated on this issue all contain gauge "smearing" terms that smear the gauge-fixing δ function (these models correspond to the Feynman gauge in gauge theories), and no spontaneous breaking of topological symmetries has yet been found [2]. (An exception might be the symmetry breaking attributed to an instanton in a noncompact base manifold [3]; we will only consider compact manifolds in which such instantons do not exist.) Meanwhile, on other issues, models without smearing terms (corresponding to the Landau gauge), owing to their simplicity, have been extensively studied as substitutes to models with smearing terms [2–6]. The presumed interchangeability of the two classes of models is based on the argument that the only difference between them—a gauge smearing term which is BRST (Becchi-Rouet-Stora-Tyutin) exact—does not affect any of the properties of the theories [1].

However, this is not unconditionally true. In fact, if Gribov zero modes [7] occur, a model in the Landau gauge may well not be connected to a model in the Feynman gauge by a smooth gauge transformation [7,8]. To see this [8], let $F(X)$ be the gauge-fixing function in the Landau gauge. Its presence in the action picks out a configuration X that is a solution of $F(X)=0$. We now try to transfer to the Feynman gauge by replacing the $F(X)=0$ by $F(X)=P$, where P is arbitrary and depends locally on the coordinates of the base space in which X lives. If the path integration does not depend on P, the Feynman gauge is arrived at by averaging over the gauge function with weight $\exp(-\alpha P^2)$. However, such P independence is contingent upon the existence, for every pair $(P,\delta P)$, of an associated infinitesimal gauge transformation v that will transform the solution to $F(X)=P$ to the solution to

$$F(X)=P+\left(\frac{\delta F}{\delta X}\frac{\delta X}{\delta v}\right)v\equiv P+\delta P\,.$$

Obviously, v may not always exist if $(\delta F/\delta X)(\delta X/\delta v)$ has zero modes (Gribov zero modes). In that case, one can no longer go from the Landau gauge to the Feynman gauge without changing physics. We are thus motivated to investigate whether topological symmetry in models in the Landau gauge could be spontaneously broken as a result of the presence of Gribov zero modes. It turns out that it can, and the main purpose of this paper is to show how this happens.

Generally, the criterion for the symmetry being spontaneously broken is that its generator Q acting on a globally well defined quantity G has a nonvanishing expectation value. In the present context, we adopt the usual approach of realizing a topological theory by a BRST procedure [9], so that Q is just the BRST operator. We study the symmetry breaking mechanism by using the method, well known in ferromagnetic theory, of adding a symmetry breaking term ϵI_1 to the topological action and examining the vacuum expectation value $\langle Q(G)\rangle$ when the proportional constant ϵ is small. Normally, an investigation of the Hilbert space including a direct analysis of the vacua would show that $\lim_{\epsilon\to 0}\langle Q(G)\rangle\neq 0$ is a criterion for the occurrence of spontaneous symmetry breaking. We assume this to be true for the present case. In this paper, we first gain an understanding of how this mechanism works in a zero-dimensional model. Then a one-dimensional supersymmetric model [6] is studied in detail. The relation of this mechanism with the existence of reducible configurations is then briefly discussed.

Given Witten's demonstration that any BRST-exact term has zero expectation value [1], and that models with and without smearing terms differ only by a BRST-exact term, one might wonder how spontaneous symmetry breaking could happen in the setting described above. The point is that when there are Gribov zero modes, $\langle Q(G)\rangle$ could be indeterminant for some G. Such an indeterminant is the reflection of an indeterminant vacuum. To remedy this indeterminant one needs to select a specific vacuum; then spontaneous symmetry breaking could be expected to occur. Consider a theory with such an indeterminant. When the indeterminant is associated with the gauge-fixing δ function, it is customarily dealt with by the insertion of a smearing term. This regularization process amounts to selecting a vacuum that preserves Q exactness [1] and yields a theory with unbroken topological symmetry [6]. Alternatively, and this is the route we shall explore in this paper, one can proceed by adding to the action a symmetry breaking term ϵI_1 whose effect will also be to pick out a specific vacuum, thereby defining a well-behaved theory with broken symmetry. To ascertain that the broken symmetry is inherent to the vacuum but not caused by the presence of the added term, one must evaluate $\langle Q(G)\rangle$ in the limit $\epsilon\to 0$. If $\langle Q(G)\rangle$ vanishes with ϵ for all G, then the vacuum preserves topological symmetry. On the other hand, if there exists some G such that $\lim_{\epsilon\to 0}\langle Q(G)\rangle\neq 0$, then topological symmetry is spontaneously broken by the vacuum. Here I_1 only serves the purpose of selecting a vacuum. Of course, there may be more than one such vacuum.

To understand this idea more concretely, we first consider a zero-dimensional model that will exhibit all the main features of the idea. Let X,B be commuting variables (bosons) and Ψ,Φ be anticommuting variables (fermions). The gauge-fixed action is $I=Q(\Psi F(X))=BF(X)+\Psi F'\Phi$; the BRST operator Q is defined by $Q(X)=-\Phi$, $Q(\Phi)=0$, $Q(\Psi)=B$, $Q(B)=0$; $F(X)$ is the gauge-fixing function; and the prime denotes derivative with respect to X. Note the absence of the gauge smearing term in the action. Let I_1 be a BRST noninvariant action and ϵ be a small parameter. The expectation value of $Q(G)$ in the presence of ϵI_1 is

$$\langle Q(G)\rangle_\epsilon=\int dX\,dB\,d\Psi\,d\Phi\,Q(G)\exp[i(I+\epsilon I_1)]\,. \quad (1)$$

If ϵ is set to zero from the outset, then, by direct expansion, one could formally show that $\langle Q(G)\rangle_0$ would be zero for any G. Nevertheless, we wish to examine the behavior of $\langle Q(G)\rangle_\epsilon$ by taking ϵ to zero *after* the expectation value has been computed. We shall show that under certain conditions $\langle Q(G)\rangle\equiv\lim_{\epsilon\to 0}\langle Q(G)\rangle_\epsilon\neq\langle Q(G)\rangle_0$. Let

$$I_1=Bf-\Psi f'\Phi\,, \quad (2)$$

where f is a (bounded) function of X. The opposite sign between the two terms on the right-hand side of (2) makes I_1 BRST noninvariant. Owing to the conservation of ghost number we only need to consider $G=\Psi H(X)$, where H is an arbitrary (bounded) function. Then

$$\langle Q(\Psi H)\rangle_\epsilon=\int dX\,dB\,d\Psi\,d\Phi(BH+\Psi H'\Phi)\exp[iB(F+\epsilon f)+i\Psi(F'-\epsilon f')\Phi]$$
$$=\int dX\,\delta(F+\epsilon f)\left[2\epsilon f'H'/(F'+\epsilon f')+H\frac{d[2\epsilon f'/(F'+\epsilon f')]}{dX}\right]. \quad (3)$$

The δ function in (3) comes from the integration over B after B in the first term of $Q(\Psi H)$ has been replaced by a derivative with respect to $F+\epsilon f$. Observe that the quantity inside the large parentheses will vanish in the limit $\epsilon\to 0$ provided F' does not vanish simultaneously with ϵ. On the other hand, the presence of the aforementioned δ function dictates that the integrand takes value only at points X that are solutions of $F+\epsilon f=0$. Thus, so long as F' and F do not vanish simultaneously with ϵ, $\lim_{\epsilon\to 0}\langle Q(\Psi H)\rangle_\epsilon=0$ and Witten's conclusion [1] that there is no spontaneous breaking of symmetry stands.

However, if F and F' vanish simultaneously at some point, say, X_c (this is the degenerate case referred to in [6]), then the integrand in (3) will in general not vanish in the limit $\epsilon\to 0$, and a nonzero integral may be expected. Notice that in this case one may not directly put F' equal to zero in the expression $2\epsilon f'/(F'+\epsilon f')$ in (3), because the argument in the δ function has been shifted by ϵf so that the function needs to be evaluated at the shifted position.

This can be done by expanding all quantities in powers of ϵ about X_c. Suppose the argument of the δ function vanishes at X_c+x_c; then to $O(\epsilon)$

$$\tfrac{1}{2}F''(X_c)x_c^2+\epsilon f(X_c)=0,\quad\text{when } f(X_c)\neq 0\,, \quad (4a)$$

$$\tfrac{1}{2}F''(X_c)x_c+\epsilon f'(X_c)=0,\quad\text{when } f(X_c)=0\,. \quad (4b)$$

Here one has to deal with these two cases separately because of their different small-x behaviors. One sees that as long as $F''(X_c)$ and $H(X_c)$ do not vanish, the term proportional to H dominates the integrand in (3):

$$\delta(F+\epsilon f)H\frac{d}{dx}\left[\frac{2\epsilon f'}{F'+\epsilon f'}\right]=-\sum_c\delta(X-X_c-x_c)$$
$$\times 2\epsilon f'(X_c)F''(X_c)|F''(X_c)x_c+\epsilon f'(X_c)|^{-3}H(X_c)+(\text{less divergent terms})\,. \quad (5)$$

Upon solving for X in (4a) and (4b) and substituting them into (5), one sees that in both cases $\langle Q(\Psi H)\rangle\sim\epsilon^{-2}$ with nonvanishing coefficient, so that it approaches infinity, rather than zero, in the limit $\epsilon\to 0$. In other words, when $F(X)$ and $F'(X)$ vanish simultaneously at X_c, it is possible that some $Q(\Psi H)$ have nonvanishing expectation value, so that the criterion for spontaneous breaking of symmetry is met. There may be more than one such X_c. Since H is an arbitrary function, its values for different X_c's are not correlated. Therefore one need not worry about the possible cancellation among different $\langle Q(\Psi H)\rangle$'s.

The infinity of $\langle Q(\Psi H)\rangle$ suggests the need for a renormalization procedure. An alternative way to obtain a finite value may be formulated as follows. Note that the choice of (2) as the symmetry breaking term is not unique. One may, for example, replace the action I_1 in (2) by a more general

$$I_1 = Bf_1(X,\epsilon) + \Psi f_2'(X,\epsilon)\Phi, \quad f_1'(X,\epsilon) \neq f_2'(X,\epsilon), \tag{6}$$

where the second condition, holding for sufficiently small ϵ, is necessary to the symmetry breaking requirement. It can be shown that if I_1 is of this form, then, with the exception of pathological choices of f_1 and f_2, $\langle Q(\Psi H)\rangle$ will be nonvanishing. For our purpose we choose them in such a way that, specifying to (4b), $f_1'(X_c,\epsilon) = f_2'(X_c,\epsilon) + \epsilon f_3(X_c) + O(\epsilon^2)$, and $H(X_c)=0$, $H'(X_c)\neq 0$ for each zero X_c of F. It follows directly that $\langle Q(\Psi H)\rangle = \sum_{X_c}[2f_3/(f_1')^2]H'|_{X_c}$, which is finite. In this case, it would be reasonable to assume that there should be a renormalization procedure that will yield the result given by this choice of I_1.

Another reason to choose I_1 as given above is the following. The existence of X_c is exactly the condition for the original partition function $Z_0 = \langle 1\rangle_0$ to be ill defined. The integrand of Z_0 is proportional to $F'/|F'|$ evaluated at the zeros of F. It becomes ill defined when F' also vanishes at at least one of these zeros, which renders the vacuum unspecified. The vacuum is specified after an I_1 is chosen. Since the Hilbert spaces for path-connected vacua are isomorphic so that the vacuum energies must be degenerate (otherwise one would be able to specify the vacuum by choosing the one having the minimum energy), the partition function Z, unlike other physical observables, has the same value for different vacua. This property imposes a very stringent restriction on the choice of I_1. It requires that, in the notation of (6), I_1 must be such that f_1 and f_2 may differ only at the $O(\epsilon)$ level. This condition coincides with the requirement for finiteness described above. Note that with I_1 so chosen, the limit $\epsilon\to 0$ becomes a double scaling limit: $\epsilon\to 0$ and $f_1'\to f_2'$. The significance of this double scaling limit is yet to be clarified.

The above discussion is meaningful only when the model has nontrivial dynamics. A simple case is supersymmetric quantum mechanics in one dimension, which we shall refer to as the one-dimensional model. In this model [6], the BRST operator is defined by $Q(X^i) = -\Phi^i$, $Q(\Phi^i)=0$, $Q(\Psi_i) = iB_i + \Psi_j\Gamma^j{}_{ik}\Phi^k$, $Q(B_i) = -B_j\Gamma^j{}_{ik}\Phi^k + (i/2)R_{ik}{}^j{}_i\Psi_j\Phi^i\Phi^k$, and the action is

$$I = -iQ\left(\oint dt[i\Psi_i(\dot X^i + F^i) + \alpha B_i\Psi_j g^{ij}]\right)$$

$$= \oint dt\{[iB_i(\dot X^i + F^i) + \Psi_i\mathcal{D}^i_j(F)\Phi^j]$$

$$+ \alpha(B_iB_jg^{ij} + \tfrac{1}{2}R^{ij}{}_{kl}\Phi^k\Phi^l\Psi_i\Psi_j)\}, \tag{7}$$

where $\mathcal{D}^i_j(F;t) = \delta^i_j(d/dt) + \dot X^k\Gamma^i_{jk} + D_jF^i$, D_k is the Riemannian covariant derivative, and the overdot means derivative with respect to the parameter t with domain S^1. Note that I is what we call "without smearing" only when $\alpha=0$. The operator T discussed in the second paragraph is here specified by $\mathcal{D}^i_j$, whose zero modes, if any, are precisely the Gribov zero modes in this model. Consider now the case $\alpha=0$. We want to see under what condition would $\langle Q(G)\rangle$ be nonvanishing for some G.

Since the action is analogous to the one in the zero-dimensional model if we identify $\dot X^k + F^k$ here with $F(X)$ there, we expect the expression for $\langle Q(G)\rangle_\epsilon$ to be similar to (3). In particular, one can see that the quantity F' in (3), whose being zero at x_c was crucial to a nonvanishing $\langle Q(G)\rangle$, would be replaced by the derivative of $\dot X^k + F^k$, which is exactly $\mathcal{D}$. In order to have a nonvanishing $\langle Q(G)\rangle$, it is sufficient that $\mathcal{D}$ have zero modes at X_c, the zero of $\dot X^k + F^k$; all of the higher modes of $\mathcal{D}$ are irrelevant.

As before we add a symmetry breaking term (the choice is not unique),

$$I_1 = \oint dt[iB_if^i - \Psi_i(D_jf^i)\Phi^j], \tag{8}$$

to the action, where $f^i(X)$ is a section of the bundle $\mathcal{TM}$ and is not necessarily related to the section F^i. Once again the opposite signs for the two terms in (8) guarantee the explicit violation of BRST symmetry. Let $G = \oint dt\,\Psi_iH^i(X)$. Then

$$\langle Q(\Psi H)\rangle_\epsilon = \int[\mathcal{D}X]\oint dt[iB_iH^i + \Psi_i(D_jH^i)\Phi^j]\exp(-I - \epsilon I_1), \tag{9}$$

where $[\mathcal{D}X]$ stands for the measure for all of X^i, B_i, Ψ_i, Φ^i. We are interested in $\langle Q(\Psi H)\rangle \equiv \lim_{\epsilon\to 0}\langle Q(\Psi H)\rangle_\epsilon$. Using a normal coordinate expansion [6], we derive

$$\langle Q(\Psi H)\rangle_\epsilon = \int DX\prod_k\delta(X^k(\epsilon))\frac{\det\mathcal{D}(-\epsilon)}{|\det\mathcal{D}(\epsilon)|}\left\{\oint ds\,D_iH^j(X(s))[\Delta^i_j(s,s,\epsilon) - \Delta^i_j(s,s,-\epsilon)]\right.$$

$$\left. + \oint ds\oint ds'\,H^m(X(s))\Delta^l_m(s,s',\epsilon)[\Sigma_l(s',\epsilon) - \Sigma_l(s',-\epsilon)]\right\}, \tag{10}$$

$$\Sigma_l(s,\epsilon) = \Delta^i_j(s,s,\epsilon)[\dot X^n R_{in}{}^j{}_l + D_lD_i(F^j + \epsilon f^j)](s),$$

where $\Delta^j_l(t,t',\epsilon)$ satisfies $\mathcal{D}^i_j(F+\epsilon f;t)\Delta^j_l(t,t',\epsilon) = \delta^i_k\delta(t-t')$. The ultraviolet divergence here is not harmful because it is related to the higher modes of $\mathcal{D}$. The right-hand side of (10) vanishes when ϵ goes to zero, unless $\Delta^j_l(t,t',\epsilon)$ has a first-order pole in ϵ (a careful investigation shows that Δ^i_j at most has first-order poles). The latter can happen only when $\mathcal{D}(0)$ has zero modes. The index theorem predicts there are in general no such zero modes in the one-dimensional model [6]. However, *accidental* zero modes may occur as a result of a judicious choice of the parametric functions F and f, namely,

$$\ker[\mathcal{D}(0)|_{X=X_c}]\neq 0\,, \tag{11}$$

for some X_c satisfying $A(X_c,\epsilon=0)$, where A is the coefficient of the pole term of Δ^i_j. As can be seen, this is precisely what we expect, the condition for the existence of Gribov zero modes.

We verify $\langle Q(G)\rangle\neq 0$ when the condition (11) is met. To show this, one only needs to evaluate the most divergent term. We consider the case of vanishing $f(X_c)$ [this corresponds to the case (4b) in the zero-dimensional model] and assume that there is only one zero mode on every $X_c(t)$. Then A lies completely in this one-dimensional space of zero modes:

$$A^j_i=-u^j u_i[u^m u_n(\mathcal{D}_m f^n)]^{-1}\,, \tag{12}$$

where u^i is the (only) zero mode. Assuming the second leading term in the expansion of $\Delta^i_j(t,t',\epsilon)$ to be nondegenerate (this is analogous to assuming $F''\neq 0$ in the zero-dimensional case), we finally obtain

$$\langle Q(\Psi H)\rangle_\epsilon=\epsilon^{-2}\sum_c 2u^i u^l u_j(\dot{X}^k R_{ki}{}^j{}_l+D_i D_j F^j) \times[u^m u_n(D_m f^n)]^{-2}u^i(H_i)\,. \tag{13}$$

As expected, $\langle Q(\Psi H)\rangle$ is not zero. We mention in passing that one may apply the method described after (6) for the zero-dimensional model to obtain a finite result.

Perhaps the most interesting question now is whether the expectation values of (some of) the physical observables in a topological theory would, as a result of $\langle Q(G)\rangle\neq 0$, become g^{ij} dependent. To answer this question, one could study the expectation value of the energy-momentum tensor $T_{ij}\equiv\partial(I+\epsilon I_1)/\partial g^{ij}$. A straightforward calculation would show that $\langle T_{ij}\rangle$ thus defined vanishes. (Note that in this case T_{ij} is no longer a Q-exact form.) This, however, would not be the right way to approach the problem. Since one's goal is to obtain the variation of the vacuum expectation value, the correct approach would be to take the $\epsilon\to 0$ limit *before* varying g^{ij}. In the present model, focusing on the topologically nontrivial partition function, we have, for $I_1=Bf(X,\epsilon)+\Psi g'(X,\epsilon)\Phi$,

$$\begin{aligned}\delta Z&=\delta(\lim_{\epsilon\to 0}Z_\epsilon)\\&=\delta\left[\lim_{\epsilon\to 0}\sum_c\det[\mathcal{D}(F+\epsilon f)]\,|\det[\mathcal{D}(F+\epsilon g)]|^{-1}\right].\end{aligned} \tag{14}$$

In the $\epsilon\to 0$ limit, all the higher modes cancel, while the zero modes contribute a ratio of A_f/A_g, where A_f is given by (12) and A_g is the same with g replacing f. This result can of course be made to be dependent on g_{ij}.

The generalization to two-dimensional σ models is straightforward. Since in this case the topological index of T is in general not zero, one should consider $\langle Q(G)\rangle$ in which G has the appropriate ghost number that allows it to absorb the topological zero modes. The condition for having nonvanishing $\langle Q(G)\rangle$ becomes that of the existence of accidental zero modes—zero modes which appear in pairs and are not subject to the index theorem. Actually this statement is in general true for any topological models. Thus the problem of whether spontaneous symmetry breaking can happen becomes the problem of finding such accidental zero modes.

It is important to notice that in the topological gauge theory [1] there are zero modes that are associated with the reducible connections, namely, connections that are invariant under some gauge transformations. Since these degrees of freedom cannot be removed by local gauge fixing, they give rise to Gribov zero modes [10]. Our analysis suggests that these reducible connections may lead to topological symmetry breaking in the Landau gauge. Another interesting case is the two-dimensional topological gravity [5,11], where a reducible configuration is a metric that admits Killing vectors. Since in this model the Landau gauge is a natural gauge, it seems that our approach should be used to study it. In this case a natural gauge-breaking term which may well be nonlocal in the moduli space needs to be found.

In summary, we have shown how spontaneous symmetry breaking can occur when Gribov zero modes are present in zero- and one-dimensional topological models without gauge smearing terms. Both models are bound to have no propagating degrees of freedom because of the low dimensionality. We expect such degrees of freedom to exist in the broken phases of other higher-dimensional, more realistic models. This needs to be confirmed.

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