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Geometry and representations of the quantum supergroup $\mathrm{OSP}_q(1|2n)$

H. C. Lee
Department of Physics and Center for Complex Systems, National Central University, Chungli, Taiwan, Republic of China

R. B. Zhang
Department of Pure Mathematics, University of Adelaide, Adelaide, Australia

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The quantum supergroup $\mathrm{OSP}_q(1|2n)$ is studied systematically. A Haar functional is constructed, and an algebraic version of the Peter–Weyl theory is extended to this quantum supergroup. Quantum homogeneous superspaces and quantum homogeneous supervector bundles are defined following the strategy of Connes' theory. Parabolic induction is developed by employing the quantum homogeneous supervector bundles. Quantum Frobenius reciprocity and a generalized Borel–Weil theorem are established for the induced representations. © *1999 American Institute of Physics.* [S0022-2488(99)00205-4]

I. INTRODUCTION

Quantized universal enveloping algebras of Lie superalgebras were introduced in the late 1980s[1,2] to describe the type of supersymmetries exhibited by some two-dimensional statistical mechanics models.[3] Since then these quantum superalgebras have been intensively studied, leading to the development of an extensive theory on both the structure and representations. We mention in particular that the quasi-triangular Hopf superalgebraic structure of the quantum superalgebras was investigated in Ref. 4; the representation theory of the type I quantum superalgebras, the $\mathrm{gl}(m|n)$ super Yangians and the quantum affine superalgebras with symmetrizable Cartan matrices were developed in Ref. 5. The theory of quantum superalgebras had significant impact on a range of areas of physics and mathematics. Its applications to two-dimensional integrable models in statistical mechanics and quantum field theory were extensively explored in Refs. 1 and 6 and many other publications. The application to knot theory and three-manifolds[7,8] has yielded many new topological invariants, notably, the multi-parameter generalizations of Alexander–Conway polynomials.

The associated quantum supergroups are in contrast less studied in the literature. So far only the quantum supergroup $\mathrm{GL}_q(m|n)$ has been systematically investigated.[9] In Ref. 9, the structure and representation theories of $\mathrm{GL}_q(m|n)$ were developed. The irreducible covariant and contravariant tensorial representations were studied in detail within the framework of parabolic induction, resulting in a quantum Borel–Weil theorem for these representations. The aim of this paper is to treat the $\mathrm{osp}(1|2n)$ series of quantum supergroups at generic q.

The $\mathrm{osp}(1|2n)$ series of Lie superalgebras played an important role in the study of supersymmetry on de Sitter space.[10] These Lie superalgebras, especially $\mathrm{osp}(1|32)$, also featured prominently in recent developments of string theory. An Inonu–Wigner contraction of $\mathrm{osp}(1|32)$ yields the 11-dimensional Poincaré superalgebra with two and five form central charges, which is the underlying symmetry of M theory; the superalgebra $\mathrm{osp}(1|32)$ itself also plays an important role in the theory of supermembranes.[11] From a mathematical point of view, $\mathrm{osp}(1|2n)$ is also rather exceptional amongst all the finite-dimensional simple Lie superalgebras in that its Cartan matrix is symmetrizable, and the structure of its finite-dimensional representations is completely understood. In particular, it is known that all finite-dimensional representations are completely reducible.

Many properties of $\mathrm{osp}(1|2n)$ carry over to the quantum case when q is generic. It is particularly useful to recall that the Drinfeld version of $\mathrm{U}_q(\mathrm{osp}(1|2n))$ is, algebraically, a trivial deformation of $\mathrm{U}(\mathrm{osp}(1|2n))$ in the sense of Gerstenhaber. (This fact is known to experts, and may be easily inferred from results of Ref. 12.) Therefore, *finite-dimensional representations* of $\mathrm{U}_q(\mathrm{osp}(1|2n))$ *are also completely reducible*. This remains true for the Jimbo version of $\mathrm{U}_q(\mathrm{osp}(1|2n))$ at generic q. One way to see this is through the specialization of the indeterminate of the Drinfeld algebra to a generic complex parameter; the other is through the isomorphism between $\mathrm{U}_q(\mathrm{osp}(1|2n))$ and $U_{-q}(\mathrm{so}(2n+1))$ established by a kind of Bose–Fermi transmutation.[13] There is also an interesting connection between the representation theory of $\mathrm{U}_q(\mathrm{osp}(1|2n))$ and quantum para-statistics, details on which can be found in Ref. 14.

This paper will study structural and representation theoretical properties of the quantum supergroup $\mathrm{OSP}_q(1|2n)$, and also investigate its underlying geometries. This quantum supergroup will be defined by its superalgebra of functions, which is the $\mathbf{Z}_2$-graded Hopf algebra generated by the matrix elements of the vector representation of $\mathrm{U}_q(\mathrm{osp}(1|2n))$. Two major results in the structure theory are presented, namely, the existence of a left and right integral, which will be called a quantum Haar functional, and a quantum Peter–Weyl theorem.

Corresponding to each reductive subalgebra $\mathrm{U}_q(\mathbf{k})$ of $\mathrm{U}_q(\mathrm{osp}(1|2n))$, we introduce a quantum homogeneous superspace, which is defined by specifying its superalgebra of functions $\mathcal{A}_q^{\mathbf{k}}$. A quantum homogeneous supervector bundle over the quantum homogeneous superspace is induced from any given finite-dimensional $\mathrm{U}_q(\mathbf{k})$ module. We shall show that the space of sections $\Gamma_q^{\mathbf{k}}(V)$ of this bundle is projective and is of finite type both as a left and a right module over $\mathcal{A}_q^{\mathbf{k}}$. Therefore our definition of quantum homogeneous supervector bundles is consistent with the general definition of noncommutative vector bundles in Connes' theory.[15]

Quantum homogeneous supervector bundles will be applied to develop a theory of induced representations for $\mathrm{OSP}_q(1|2n)$. Amongst the results obtained are quantum versions of Frobenius reciprocity and the Borel–Weil theorem. The latter provides a concrete realization of finite-dimensional irreducible $\mathrm{OSP}_q(1|2n)$ representations in terms of quantum analogs of ''holomorphic'' sections of quantum homogeneous supervector bundles.

We wish to point out that in the context of Lie supergroups at the classical level, the mathematical theories of homogeneous superspaces and homogeneous supervector bundles were studied in Refs. 16 and 17. The development of a Bott–Borel–Weil theory was also initiated and extensively investigated by Penkov and co-workers.[17] However, complications arising from supermanifold geometry render these subjects very difficult to study. So far as we are aware, many aspects of the subjects remain to be fully developed. It seems that the Hopf algebraic approach developed here and in Ref. 9 is also worth exploring at the classical level, and is likely to provide a new method complementary to the geometric approach of Refs. 16 and 17.

The organization of the paper is as follows. In Sec. II we review some known facts about $\mathrm{U}_q(\mathrm{osp}(1|2n))$, which will be needed later. In Sec. III we study the quantum supergroup $\mathrm{OSP}_q(1|2n)$. In Sec. IV we investigate the quantum homogeneous superspaces and quantum homogeneous supervector bundles determined by this quantum supergroup, while the last section applies results of Sec. IV to study the representation theory of $\mathrm{OSP}_q(1|2n)$.

II. $\mathrm{U}_q(\mathrm{osp}(1|2n))$

This section reviews some known results on the quantized universal enveloping algebra $\mathrm{U}_q(\mathrm{osp}(1|2n))$. Let E be the n-dimensional Euclidean space spanned by the vectors ϵ_i, with the inner product (,) defined by $(\epsilon_i,\epsilon_j)=\delta_{ij}$. We can express the simple roots of the Lie superalgebra $\mathrm{osp}(1|2n)$ in terms of the ϵ's as

$$\alpha_i=\epsilon_i-\epsilon_{i+1}, \quad i=1,2,...,n-1, \quad \alpha_n=\epsilon_n,$$

where α_n is the odd simple root. The Cartan matrix $A=(a_{ij})_{i,j=1}^n$ of $\mathrm{osp}(1|2n)$ is then given by $a_{ij}=2(\alpha_i,\alpha_j)/(\alpha_i,\alpha_i)$. An element $\mu\in E$ will be called integral if

$$l_i = \frac{2(\mu,\alpha_i)}{(\alpha_i,\alpha_i)} \in \mathbf{Z}, \quad \forall i<n, \quad l_n = \frac{(\mu,\alpha_n)}{(\alpha_n,\alpha_n)} \in \mathbf{Z},$$

and the set of all integral elements will be denoted by $\mathcal{P}$. (Note the unusual form of l_n.) Set $\mathcal{P}_+ = \{\mu \in \mathcal{P} | l_i, l_n \in \mathbf{Z}_+\}$. Elements of $\mathcal{P}_+$ will be called integral dominant.

The Jimbo version of the quantum superalgebra $\mathrm{U}_q(\mathrm{osp}(1|2n))$ is a $\mathbf{Z}_2$-graded complex associative algebra generated by $\{k_i^{\pm 1}, e_i, f_i, i \in \mathbf{N}_n\}$, $\mathbf{N}_n = \{1,2,...,n\}$, subject to the relations

$$\begin{gathered} k_i k_i^{-1} = 1, \quad k_i k_j = k_j k_i, \\ k_i e_j = q^{(\alpha_i,\alpha_j)} e_j k_i, \quad k_i f_j = q^{-(\alpha_i,\alpha_j)} f_j k_i, \\ [e_i, f_j\} = \delta_{ij} \frac{k_i - k_i}{q - q}, \quad \forall i,j \in I, \\ (\mathrm{Ad}\, e_i)^{1-a_{ij}}(e_j) = 0, \quad (\mathrm{Ad}\, f_i)^{1-a_{ij}}(f_j) = 0, \quad \forall i \neq j. \end{gathered} \tag{1}$$

All the generators are chosen to be homogeneous, with $k_i^{\pm 1}$, $\forall i$, and e_j, f_j, $j<n$, being even, and e_n, f_n being odd. For a homogeneous element x, we define $[x]=0$ if x is even, and $[x]=1$ when odd. The graded commutator $[.,.\}$ represents the usual commutator when any one of the two arguments is even, and the anticommutator when both arguments are odd. The adjoint operation Ad is defined by

$$\begin{gathered} \mathrm{Ad}\, e_i(x) = e_i x - (-1)^{[e_i][x]} k_i x k_i^{-1} e_i, \\ \mathrm{Ad}\, f_i(x) = f_i x - (-1)^{[f_i][x]} k_i x k_i^{-1} f_i. \end{gathered}$$

For x being a monomial in e_j's or f_j's it carries a definite weight $\omega(x) \in H^*$. Then $\mathrm{Ad}\, e_i(x) = e_i x - (-1)^{[e_i[x]} q^{(\alpha_i,\omega(x))} x e_i$, and similarly for $\mathrm{Ad}\, f_i(x)$. For convenience, we will use the notation $\mathbf{g}$ to denote $\mathrm{osp}(1|2n)$, and $\mathrm{U}_q(\mathbf{g})$ to denote $\mathrm{U}_q(\mathrm{osp}(1|2n))$. As is well known, this algebra has the structures of a $\mathbf{Z}_2$-graded Hopf algebra. We will denote the comultiplication by Δ, the counit by ϵ, and the antipode by S.

The representation theory of $\mathrm{U}_q(\mathbf{g})$ was developed in Ref. 13. For any finite-dimensional $\mathrm{U}_q(\mathbf{g})$ module, there exists a homogeneous basis relative to which the k_i are represented by diagonal matrices. Here we will only consider such finite-dimensional $\mathrm{U}_q(\mathbf{g})$ modules that the eigenvalues of the k_i tend to 1 as q approaches 1. We will denote the set of all such $\mathrm{U}_q(\mathbf{g})$ modules by $\mathbf{Mod}_q(\mathbf{g})$. Recall that all objects of $\mathbf{Mod}_q(\mathbf{g})$ are semi-simple.

If $W(\lambda)$ is a simple object of $\mathbf{Mod}_q(\mathbf{g})$, then there exists the unique (up to scalar multiples) highest weight vector v_+, such that

$$e_i v_+ = 0, \quad k_i v_+ = q^{(\lambda,\alpha_i)} v_+, \quad \lambda \in \mathcal{P}_+,$$

and the module $W(\lambda)$ is uniquely determined by the highest weight λ. We will denote the lowest weight of $W(\lambda)$ by $\overline{\lambda}$, and define $\lambda^\dagger = -\overline{\lambda}$. The dual module of $W(\lambda)$ has highest weight $\lambda^\dagger$.

The irreducible $\mathrm{U}_q(\mathbf{g})$ module with highest weight ϵ_1 plays a special role in the representation theory of $\mathrm{U}_q(\mathbf{g})$. We denote this module by $\mathbf{E}$, and refer to it as the vector module. Let us now examine this module in some detail. Denote by w_1 the highest weight vector of $\mathbf{E}$, which is assumed to be even. Define

$$\begin{gathered} w_i = f_{i-1} w_{i-1}, \quad 1 < i \leqslant n, \\ w_0 = f_n w_n, \quad w_{-n} = f_n w_n, \\ w_{-j} = f_j w_{-j-1}, \quad n > j \geqslant 1. \end{gathered}$$

Then $\{w_\mu | \mu=0,\pm 1,\pm 2,...,\pm n\}$ forms a weight basis of $\mathbf{E}$. We will denote by t the irreducible representation relative to this basis. The matrix elements of the e_i, f_i and k_i can be immediately written down. We have

$$t(e_i)_{\mu\nu}=\delta_{\mu i}\delta_{\nu,i+1}+\delta_{\mu,-i-1}\delta_{\nu,-i},$$

$$t(f_i)_{\mu\nu}=\delta_{\mu,i+1}\delta_{\nu i}+\delta_{\mu,-i}\delta_{\nu,-i-1}, \quad i<n,$$

$$t(e_n)_{\mu\nu}=\delta_{\mu n}\delta_{\nu 0}-\delta_{\mu 0}\delta_{\nu,-n},$$

$$t(f_n)_{\mu\nu}=\delta_{\mu 0}\delta_{\nu n}+\delta_{\mu,-n}\delta_{\nu 0},$$

$$t(k_j)_{\mu\nu}=\delta_{\mu\nu}q^{(\alpha_j,\epsilon_\mu)}, \quad 1\leq j\leq n,$$

where $\epsilon_0=0$, and $\epsilon_{-i}=-\epsilon_i$.

Let $\{w_\mu^*\}$ be the basis of $\mathbf{E}^*$ defined by $w_\mu^*(w_\nu)=\delta_{\mu\nu}$. Here $\mathbf{E}^*$ has a natural $\mathrm{U}_q(\mathbf{g})$-module structure with the $\mathrm{U}_q(\mathbf{g})$ action given by

$$xw_\mu^*=\sum_\nu (-1)^{[x]\delta_{\mu 0}}t(S(x))_{\mu\nu}w_\nu^*. \tag{2}$$

The lowest weight of $\mathbf{E}$ is $-\epsilon_1$. Thus the module $\mathbf{E}$ is self-dual. This implies that there exists a $\mathrm{U}_q(\mathbf{g})$-module isomorphism $M:\mathbf{E}\rightarrow\mathbf{E}^*$, which is unique up to scaler multiples. The w_{-1}^*, being the highest weight vector of $\mathbf{E}^*$, will be identified with w_1 so that this arbitrariness in M can be removed. Now let

$$w_\mu^*=\sum_\nu w_\nu M_{\nu\mu}.$$

Then

$$M_{\mu\nu}=m_\mu\delta_{\mu+\nu,0}, \quad m_\mu=\begin{cases}(-q)^{\mu-1}, & \mu>0,\\ (-q)^n, & \mu=0,\\ (-q)^{2n+\mu}, & \mu<0.\end{cases} \tag{3}$$

It follows from earlier discussions that repeated tensor products of $\mathbf{E}$ are completely reducible. Furthermore, every finite-dimensional irreducible $\mathrm{U}_q(\mathbf{g})$ module is embedded in some $\mathbf{E}^{\otimes k}$ for at least one $k\geq 0$.

For later use, we consider two classes of $\mathbf{Z}_2$-graded Hopf subalgebras of $\mathrm{U}_q(\mathbf{g})$. Corresponding to any subset Θ of $\mathbf{N}_n$, we introduce

$$\mathcal{S}_k=\{k_i^{\pm 1}, i\in\mathbf{N}_n; \quad e_j, f_j, j\in\mathbf{\Theta}\};$$

$$\mathcal{S}_p=\mathcal{S}_k\cup\{e_j, j\in\mathbf{N}_n\backslash\mathbf{\Theta}\}.$$

The elements of each set generate a $\mathbf{Z}_2$-graded Hopf subalgebra of $\mathrm{U}_q(\mathbf{g})$. The subalgebra generated by the elements of $\mathcal{S}_k$ will be denoted by $\mathrm{U}_q(\mathbf{k})$, and called a reductive subalgebra of $\mathrm{U}_q(\mathbf{g})$, while that generated by the elements of $\mathcal{S}_p$ will be denoted by $\mathrm{U}_q(\mathbf{p})$ and called a parabolic subalgebra. Note that $\mathrm{U}_q(\mathbf{k})$ is a $\mathbf{Z}_2$-graded Hopf subalgebra of $\mathrm{U}_q(\mathbf{p})$. If we replace e_i by f_i and vice versa in $\mathcal{S}_p$, we obtain another set, which will generate a $\mathbf{Z}_2$-graded Hopf subalgebra of $\mathrm{U}_q(\mathbf{g})$ having similar properties as $\mathrm{U}_q(\mathbf{p})$. Results presented in the remainder of the paper can also be formulated using such algebras.

Observe that there are two types of reductive subalgebras, depending on whether $\mathbf{\Theta}$ contains n. The first type arises when $n\notin\Theta$, and in this case, $\mathrm{U}_q(\mathbf{k})$ is the direct product of quantized universal enveloping algebras associated with a series of ordinary (i.e., nongraded) Lie algebras of type A supplemented by the algebra generated by some $k_i^{\pm 1}$. The second type arises when $n \in \Theta$. This time, $\mathrm{U}_q(\mathbf{k})$ is the direct product of the first type with a $\mathrm{U}_q(\mathrm{osp}(1|2m))$ for some $m < n$. In both cases, the finite-dimensional representations of $\mathrm{U}_q(\mathbf{k})$ are completely reducible. This fact will be of great importance to the main subject of the paper.

Let V_μ be a finite-dimensional irreducible $\mathrm{U}_q(\mathbf{k})$ module. Then V_μ is of highest weight type. Let μ be the highest weight and $\tilde{\mu}$ the lowest weight of V_μ respectively. We can extend V_μ in a unique fashion to a $\mathrm{U}_q(\mathbf{p})$ module, which is still denoted by V_μ, such that the elements of $\mathcal{S}_p \backslash \mathcal{S}_k$ act by zero. It is not difficult to see that all finite dimensional irreducible $\mathrm{U}_q(\mathbf{p})$ modules are of this kind.

Consider a finite-dimensional irreducible $\mathrm{U}_q(\mathbf{g})$ module $W(\lambda)$, with highest weight λ and lowest weight $\bar{\lambda}$. $W(\lambda)$ can be restricted in a natural way to a $\mathrm{U}_q(\mathbf{p})$ module, which is always indecomposable, but not irreducible in general. It can be readily shown that

$$\dim_{\mathbf{C}} \mathrm{Hom}_{\mathrm{U}_q(\mathbf{p})}(W(\lambda), V_\mu) = \begin{cases} 1, & \bar{\lambda} = \tilde{\mu}, \\ 0, & \bar{\lambda} \neq \tilde{\mu}. \end{cases}$$

III. THE QUANTUM SUPERGROUP $\mathrm{OSP}_q(1|2n)$

There exist well-established methods for quantizing ordinary Lie groups in the non-supersymmetric setting. (See Ref. 18 and references therein.) These methods can also be extended to construct $\mathrm{OSP}_q(1|2n)$, and this will be done here. However, we should point out that it is, in general, much more difficult to study quantum supergroups. See Ref. 9 for details on $\mathrm{GL}_q(m|n)$.

We will show that the quantum supergroup $\mathrm{OSP}_q(1|2n)$ admits a quantum Haar functional, and also a Peter–Weyl basis. This, however, is an exception rather than the rule. It is known that the finite-dimensional representations of all the quantum superalgebras but $\mathrm{U}_q(\mathrm{osp}(1|2n))$ are not completely reducible. This fact renders it impossible to construct Peter–Weyl bases for the corresponding quantum supergroups [which are yet to be defined except $\mathrm{GL}_q(m|n)$].

Let us recall some general results about $\mathbf{Z}_2$-graded Hopf algebras. Let A be a $\mathbf{Z}_2$-graded Hopf algebra with comultiplication Δ, counit ϵ, and antipode S. We define the finite dual A^0 of A to be a subspace of A^* such that for any $f \in A^0$, $\mathrm{Ker}\, f$ contains a two-sided ideal $\mathcal{I}$ of A which is of finite codimension, i.e., $\dim A/\mathcal{I} < \infty$. Of course in the most general situation, there is no guarantee that A^0 will not be zero. But when A^0 is nontrivial, then it is also a $\mathbf{Z}_2$-graded Hopf algebra with a structure dualizing that of A. More explicitly, the multiplication is defined, for f, $g \in A^0$, a, $b \in A$, by

$$\langle fg, a \rangle = \langle f \otimes g, \Delta(a) \rangle = \sum_{(a)} (-1)^{[g][a_{(1)}]} \langle f, a_{(1)} \rangle \langle g, a_{(2)} \rangle.$$

It is easy to see that the unit of A^0 is ϵ. Denote the comultiplication, the counit, and the antipode of A^0 respectively by Δ_0, ϵ^0 and S_0. Then

$$\langle \Delta_0(f), a \otimes b \rangle = \sum_{(f)} (-1)^{[f_{(1)}][f_{(2)}]} \langle f_{(1)}, a \rangle \langle f_{(2)}, b \rangle = \langle f, ab \rangle,$$

$$\langle S_0(f), a \rangle = \langle f, S(a) \rangle, \quad \epsilon^0(f) = \langle f, \mathbb{1}_A \rangle.$$

Now we come back to the quantum supergroup $\mathrm{OSP}_q(1|2n)$. As is well known, we cannot define the quantum supergroup directly. Instead, we need to find the algebra of functions on it. Introduce $t_{\mu\nu} \in (\mathrm{U}_q(\mathbf{g}))^*$, $\mu, \nu = 0, \pm 1, \pm 2, \ldots, \pm n$, defined by

$$t_{\mu\nu}(x) = t(x)_{\mu\nu}, \quad \forall x \in \mathrm{U}_q(\mathbf{q}),$$

where t is the vector representation of $U_q(\mathbf{g})$. We call the $t_{\mu\nu}$ the matrix elements of t. Finite dimensionality of $\mathbf{E}$ implies that $t_{\mu\nu}\in(U_q(\mathbf{g}))^0$, $\forall\mu$, ν.

We define the superalgebra $\mathcal{T}_q(\mathbf{g})$ of functions on $OSP_q(1|2n)$ to be the $\mathbf{Z}_2$-graded subalgebra of $(U_q(\mathbf{g}))^0$ generated by the matrix elements of the vector representation of $U_q(\mathbf{g})$, i.e., $t_{\mu\nu}$, $\mu,\nu=0,\pm 1,\pm 2,...,\pm n$. Then we have the following theorem.

Theorem 1: (1) *$\mathcal{T}_q(\mathbf{g})$ is a $\mathbf{Z}_2$-graded Hopf algebra.*

(2) *Let $t^{(\lambda)}$ be the irreducible representation of $U_q(\mathbf{g})$ with highest weight $\lambda\in\mathcal{P}_+$, and let $t^{(\lambda)}_{ij}$, $i,j=1,2,...,d_\lambda$ ($d_\lambda=\dim t^{(\lambda)}$), be the matrix elements of $t^{(\lambda)}$. Then*

$$\mathcal{T}_q(\mathbf{g})=\bigoplus_{\lambda\in\mathcal{P}_+}\bigoplus_{i,j=1}^{d_\lambda}\mathbf{C}t^{(\lambda)}_{ij}. \tag{4}$$

Proof: The $\mathbf{Z}_2$-graded bialgebra structure of $\mathcal{T}_q(\mathbf{g})$ is obvious, and the existence of the antipode follows from the self-duality of the vector module $\mathbf{E}$ over $U_q(\mathbf{g})$. Part (2) immediately follows from the complete reducibility of finite-dimensional representations of $U_q(\mathbf{g})$. □

Let us now work out the explicit forms of the comultiplication and the antipode. The comultiplication is given by

$$\Delta_0(t_{\mu\nu})=\sum_\sigma(-1)^{(\delta_{\mu 0}+\delta_{\sigma 0})(\delta_{\nu 0}+\delta_{\sigma 0})}t_{\mu\sigma}\otimes t_{\sigma\nu}.$$

The antipode can be constructed from (2) by using the $U_q(\mathbf{g})$-module isomorphism M. We have

$$S_0(t_{\mu\nu})=(-1)^{(\delta_{\mu 0}+\delta_{\nu 0})\delta_{\mu 0}}(M^{-1}tM)_{\nu\mu}$$
$$=(-1)^{(\delta_{\mu 0}+\delta_{\nu 0})\delta_{\mu 0}}\frac{m_{-\mu}t_{-\nu,-\mu}}{m_{-\nu}},$$

where m_μ is given by (3).

Here we introduce more notations for later use. Let $\{w^{(\lambda)}_i|i=1,2,...,d_\lambda\}$ be the homogeneous basis of $W(\lambda)$ with respect to which the representation $t^{(\lambda)}$ is defined. We denote by $\{\tilde{w}^{(\lambda)}_i|i=1,2,...,d_\lambda\}$ the basis of $W(\lambda)^*=W(\lambda^\dagger)$ such that $\tilde{w}^{(\lambda)}_i(w^{(\lambda)}_j)=\delta_{ij}$. The $U_q(\mathbf{g})$-module structure of $W(\lambda)^*$ enables us to define $\tilde{t}^{(\lambda)}_{ij}\in\mathcal{T}_q(\mathbf{g})$ by

$$x\tilde{w}^{(\lambda)}_i=\sum_j\tilde{t}^{(\lambda)}_{ji}(x)\tilde{w}^{(\lambda)}_j,\quad \forall x\in U_q(\mathbf{g}).$$

Then

$$\tilde{t}^{(\lambda)}_{ji}=(-1)^{[i]([i]+[j])}S_0(t^{(\lambda)}_{ij}),$$

where $[i]=0$ or 1 depending on whether w_i is even or odd. Clearly the $\tilde{t}^{(\lambda)}_{ji}$ are linear combinations of $t^{(\lambda^\dagger)}_{ij}$. Furthermore, the $\tilde{t}^{(\lambda)}_{ji}$, $\forall\lambda\in\mathcal{P}_+$, also form a basis of $\mathcal{T}_q(\mathbf{g})$.

From here on, we will omit the subscript 0 from Δ_0 and S_0.

Let us now turn to the discussion of a Haar functional on the quantum supergroup $\mathcal{T}_q(\mathbf{g})$. But before embarking on this task, we first consider the notion of an integral on an arbitrary $\mathbf{Z}_2$-graded Hopf algebra A. Let A^* be its dual, which has a natural $\mathbf{Z}_2$-graded algebraic structure induced by the co-algebraic structure of A. An even homogeneous element $\int^l\in A^*$ is called a left integral on A if

$$f\cdot\int^l=\langle f,\mathbb{1}_A\rangle\int^l,\quad \forall f\in A^*.$$

Similarly, an even homogeneous element $\int^r\in A^*$ is called a right integral on A if

$$\int^r \cdot f = \langle f, \mathbb{1}_A \rangle \int^r, \quad \forall f \in A^*.$$

A straightforward calculation shows that the defining properties of the integrals are equivalent to the following requirements

$$\left(\mathrm{id} \otimes \int^l\right)\Delta(x) = \int^l x, \quad \left(\int^r \otimes \mathrm{id}\right)\Delta(x) = \int^r x, \quad \forall x \in A. \tag{5}$$

where id is the identity map on A.

A Haar functional $\int \in A^*$ on A is an integral on A which is both left and right, and sends $\mathbb{1}_A$ to 1, i.e.,

$$\begin{aligned} &(i) \quad \left(\int \otimes \mathrm{id}\right)\Delta(x) = \left(\mathrm{id} \otimes \int\right)\Delta(x) = \int x, \quad \forall x \in A, \\ &(ii) \quad \int \mathbb{1}_A = 1. \end{aligned} \tag{6}$$

In the case of $\mathcal{T}_q(\mathbf{g})$, it is an entirely straightforward matter to show the following.

Theorem 2: *The element* $\int \in (\mathcal{T}_q(\mathbf{g}))^*$ *defined by*

$$\int \mathbb{1}_{\mathcal{T}_q(\mathbf{g})} = 1; \quad \int t_{ij}^{(\lambda)} = 0, \quad 0 \neq \lambda \in \mathcal{P}_+,$$

gives rise to a Haar functional on $\mathcal{T}_q(\mathbf{g})$.

Denote by 2ρ the sum of the positive roots of $\mathbf{g}$. Let $K_{2\rho}$ be the product of powers of $k_i^{\pm 1}$'s such that

$$K_{2\rho} e_i K_{2\rho}^{-1} = q^{(2\rho,\alpha_i)} e_i, \quad \forall i.$$

Then it can be easily shown that

$$S^2(x) = K_{2\rho} x K_{2\rho}^{-1}, \quad \forall x \in \mathrm{U}_q(\mathbf{q}).$$

We define the quantum superdimension of the irreducible $\mathrm{U}_q(\mathbf{g})$ module $W(\lambda)$ by

$$\mathrm{SD}_q(\lambda) := \mathrm{Str}\{t^{(\lambda)}(K_{2\rho})\}.$$

For quantum superalgebras other than the $\mathrm{osp}(1|2n)$ series, there exists a class of finite-dimensional irreducible representations, the typicals, of which the super-dimensions vanish identically. Again, $\mathrm{U}_q(\mathrm{osp}(1|2n))$ is an exception, and we have the following important property: for any irreducible $\mathrm{U}_q(\mathrm{osp}(1|2n))$ module $W(\lambda)$ with highest weight $\lambda \in \mathcal{P}_+$,

$$\mathrm{SD}_q(\lambda) \neq 0.$$

Now the Haar functional $\int$ satisfies the following properties.

Lemma 1:

$$\begin{aligned} &\int t_{ij}^{(\lambda)} \tilde{t}_{rs}^{(\mu)} (-1)^{[j][r]+[i]+[j]} = \delta_{ir} \delta_{\lambda\mu} \frac{t_{sj}^{(\lambda)}(K_{2\rho})}{SD_q(\lambda)}, \\ &\int \tilde{t}_{ij}^{(\lambda)} t_{rs}^{(\mu)} (-1)^{[j][r]} = \delta_{js} \delta_{\lambda\mu} \frac{\tilde{t}_{ir}^{(\lambda)}(K_{2\rho})}{SD_q(\lambda)}. \end{aligned} \tag{7}$$

Proof: Consider the first equation. The $\lambda \neq \mu$ case is easy to prove: the integral vanishes because the tensor product $W(\lambda)\otimes W(\mu^\dagger)$ does not contain the trivial $U_q(\mathbf{g})$ module. When $\lambda = \mu$, we introduce the notations

$$\phi_{ir;sj} = \int t_{ij}^{(\lambda)} \tilde{t}_{rs}^{(\lambda)} (-1)^{[j][r]+[i]+[j]}; \quad \Phi[s,j] = (\phi_{ir;sj})_{i,r=1}^{d_\lambda}; \quad \Psi[i,r] = (\phi_{ir;sj})_{s,j=1}^{d_\lambda}.$$

It is clearly true that $\mathrm{Str}(\Psi[i,r]) = \delta_{ir}$.

Note that corresponding to each $x \in U_q(\mathbf{g})$, there exists an $\tilde{x} \in (\mathcal{T}_q(\mathbf{g}))^*$ defined by $\tilde{x}(a) = \langle a,x\rangle$, $\forall a \in \mathcal{T}_q(\mathbf{g})$. The left integral property of $\int$ leads to

$$\begin{aligned}\epsilon(x)\phi_{ir;sj} &= \left(\tilde{x}.\int\right) t_{ij}^{(\lambda)} \tilde{t}_{rs}^{(\lambda)} (-1)^{[j][r]+[i]+[j]} \\ &= \sum_{(x)} \sum_{i',r'} t_{ii'}^{(\lambda)}(x_{(1)}) t_{r'r}^{(\lambda)}(S(x_{(2)})) \phi_{i'r';sj} (-1)^{[x]([i]+[j])+[x_{(2)}]([j]+[s])},\end{aligned}$$

i.e.,

$$\epsilon(x)\Phi[s,j] = \sum_{(x)} t^{(\lambda)}(x_{(1)}) \Phi[s,j] t^{(\lambda)}(S(x_{(2)}))(-1)^{[x_{(2)}]([j]+[s])}, \quad \forall x \in U_q(\mathbf{g}).$$

Schur's lemma forces $\Phi[s,j]$ to be proportional to the identity matrix, and we have

$$\Psi[i,r] = \delta_{ir}\psi,$$

for some $d_\lambda \times d_\lambda$ matrix ψ. The right integral property of $\int$ leads to

$$\epsilon(y)\psi = \sum_{(y)} t^{(\lambda)}(K_{2\rho}) t^{(\lambda)}(y_{(1)}) t^{(\lambda)}(K_{2\rho}^{-1}) \psi t^{(\lambda)}(S(y_{(2)})).$$

Again by using Schur's lemma we conclude that ψ is proportional to $t^{(\lambda)}(K_{2\rho})$. Since its supertrace is 1, we have

$$\psi = \frac{t^{(\lambda)}(K_{2\rho})}{\mathrm{SD}_q(\lambda)}.$$

This completes the proof of the first equation of the lemma. The second equation can be shown in exactly the same way. □

It is worth observing that this Lemma and part (2) of Theorem 1 provide a quantum analog of the Peter–Weyl theorem for $\mathrm{OSP}_q(1|2n)$.

IV. QUANTUM HOMOGENEOUS SUPERVECTOR BUNDLES

In this section we will investigate the quantum homogeneous superspaces and quantum homogeneous supervector bundles arising from the quantum supergroup $\mathrm{OSP}_q(1|2n)$ by adapting the methods and techniques of Refs. 9 and 19 to the present context. Let us start by introducing two types of actions of $U_q(\mathbf{g})$ on $\mathcal{T}_q(\mathbf{g})$. The first action will be denoted by $\circ$, which corresponds to the right translation in the classical theory of Lie groups. It is defined by

$$x \circ f = \sum_{(f)} (-1)^{[f_{(1)}][f_{(2)}]} f_{(1)} \langle f_{(2)}, x\rangle, \quad x \in U_q(\mathbf{g}), \quad f \in \mathcal{T}_q(\mathbf{g}). \tag{8}$$

Straightforward calculations show that

$$x\circ(y\circ f)=(xy)\circ f,\quad (x\circ f)(y)=f(yx),\quad (\mathrm{id}_{\mathcal{T}_q(\mathbf{g})}\otimes x\circ)\Delta(f)=\Delta(x\circ f).$$

The other action, which corresponds to the left translation in the classical Lie group theory, will be denoted by $\cdot$. It is defined by

$$x\cdot f=\sum_{(f)}\langle f_{(1)},S^{-1}(x)\rangle f_{(2)}.\tag{9}$$

It can be easily shown that

$$(x\cdot f)(y)=(-1)^{[x][y]}f(S^{-1}(x)y),$$

$$x\cdot(y\cdot f)=(xy)\cdot f,\quad x,y\in \mathrm{U}_q(\mathbf{g}),\quad f\in\mathcal{T}_q(\mathbf{g}).$$

Furthermore, the two actions graded commute in the following sense

$$x\circ(y\cdot f)=(-1)^{[x][y]}y\cdot(x\circ f),\quad x,y\in\mathrm{U}_q(\mathbf{g}),\quad f\in\mathcal{T}_q(\mathbf{g}).$$

Let V be a finite-dimensional module over $\mathrm{U}_q(\mathbf{k})$. We extend the actions $\circ$ and $\cdot$ trivially to $V\otimes\mathcal{T}_q(\mathbf{g})$: for any $\zeta=\Sigma v_i\otimes f_i\in V\otimes\mathcal{T}_q(\mathbf{g})$,

$$x\cdot\zeta=\sum(-1)^{[x][v_i]}v_i\otimes x\cdot f_i,$$

$$x\circ\zeta=\sum(-1)^{[x][v_i]}v_i\otimes x\circ f_i,\quad x\in\mathrm{U}_q(\mathbf{g}).$$

We now introduce two important definitions:

$$\mathcal{A}_q^{\mathbf{k}}:=\{f\in\mathcal{T}_q(\mathbf{g})\,|\,x\circ f=\epsilon(x)f,\quad\forall x\in\mathrm{U}_q(\mathbf{k})\};\tag{10}$$

$$\Gamma_q^{\mathbf{k}}(V):=\{\zeta\in V\otimes\mathcal{T}_q(\mathbf{g})\,|\,x\circ\zeta=(S(x)\otimes\mathrm{id}_{\mathcal{T}_q(\mathbf{g})})\zeta,\quad\forall x\in\mathrm{U}_q(\mathbf{k})\}.\tag{11}$$

The remainder of this section is devoted to studying the properties of these objects. Let us first prove the following.

Proposition 1: (1) $A_q^{\mathbf{k}}$ *is an infinite-dimensional subalgebra of* $\mathcal{T}_q(\mathbf{g})$.

(2) $\Gamma_q^{\mathbf{k}}(V)$ *is an infinite-dimensional supervector space if the weight of any vector of V is* $\mathrm{U}_q(\mathbf{g})$ *integral, and is zero otherwise*.

Proof: We first show that $\mathcal{A}_q^{\mathbf{k}}$ is a subalgebra of $\mathcal{T}_q(\mathbf{g})$. Since $\mathrm{U}_q(\mathbf{k})$ is a Hopf subalgebra of $\mathrm{U}_q(\mathbf{g})$, for any $x\in\mathrm{U}_q(\mathbf{k})$, $\Delta(x)=\Sigma_{(x)}x_{(1)}\otimes x_{(2)}\in\mathrm{U}_q(\mathbf{k})\otimes\mathrm{U}_q(\mathbf{k})$. Hence

$$x\circ(ab)=\sum_{(x)}(-1)^{[x_{(2)}][a]}\{x_{(1)}\circ a\}\{x_{(2)}\circ b\}=\epsilon(x)ab,$$

that is, $ab\in\mathcal{A}_q^{\mathbf{k}}$.

Since the finite-dimensional representations of $\mathrm{U}_q(\mathbf{k})$ are completely reducible, the study of properties of $\Gamma_q^{\mathbf{k}}(V)$ reduces to the case when V is irreducible. Let V_μ be a finite-dimensional irreducible $\mathrm{U}_q(\mathbf{k})$ module with highest weight μ and lowest weight $\tilde{\mu}$. Any element $\zeta\in\Gamma_q^{\mathbf{k}}(V_\mu)$ can be expressed in the form

$$\zeta=\sum_{\lambda\in\mathcal{P}_+}\sum_{i,j}v_{ij}^{(\lambda)}\otimes\tilde{t}_{ij}^{(\lambda)},$$

for some $v_{ij}^{(\lambda)} \in V_\mu$. Fix an arbitrary $\lambda \in \mathcal{P}_+$. For any nonvanishing $w \in W(\lambda)$, the following linear map is clearly surjective:

$$\mathrm{Hom}_{\mathbf{C}}(W(\lambda), V_\mu) \otimes w \to V_\mu,$$

$$\phi \otimes w \mapsto \phi(w).$$

Thus there exist $\phi_i^{(\lambda)} \in \mathrm{Hom}_{\mathbf{C}}(W(\lambda), V_\mu)$ such that $v_{ij}^{(\lambda)} = \phi_i^{(\lambda)}(w_j^{(\lambda)})$, where $\{w_i^{(\lambda)}\}$ is the basis of $W(\lambda)$ discussed before. Therefore, we can rewrite ζ as

$$\zeta = \sum_{\lambda \in \mathcal{P}_+} \sum_{i,j} \phi_i^{(\lambda)}(w_j^{(\lambda)}) \otimes \tilde{t}_{ij}^{(\lambda)}.$$

The defining property of $\Gamma_q^{\mathbf{k}}(V_\mu)$ states that

$$\ell \circ \zeta = (\mathrm{id}_{\mathcal{T}_q(\mathbf{g})} \otimes S(\ell))\zeta, \quad \forall \ell \in \mathrm{U}_q(\mathbf{k}).$$

Thus we have

$$\sum_{\lambda \in \mathcal{P}_+} \sum_{i,j,k} t_{jk}^{(\lambda)}(S(\ell))\phi_i^{(\lambda)}(w_j^{(\lambda)}) \otimes (-1)^{[\ell][\phi_i^{(\lambda)}]} \tilde{t}_{ik}^{(\lambda)} = \sum_{\lambda \in \mathcal{P}_+} \sum_{i,j} S(\ell)\phi_i^{(\lambda)}(w_j^{(\lambda)}) \otimes \tilde{t}_{ij}^{(\lambda)}.$$

Recalling that the $\tilde{t}_{ki}^{(\lambda)}$ are linearly independent, the above is equivalent to

$$\ell \phi_i^{(\lambda)}(w_j^{\lambda}) = (-1)^{[\ell][\phi_i^{(\lambda)}]} \phi_i^{(\lambda)}(\ell w_j^{(\lambda)}), \quad \forall \ell \in \mathrm{U}_q(\mathbf{k}).$$

This equation is precisely the statement that the $\phi_i^{(\lambda)}$ be $\mathrm{U}_q(\mathbf{k})$-module homomorphisms of degrees $[\phi_i^{(\lambda)}]$,

$$\phi_i^{(\lambda)} \in \mathrm{Hom}_{\mathrm{U}_q(\mathbf{k})}(W(\lambda), V_\mu) \subset \mathrm{Hom}_{\mathbf{C}}(W(\lambda), V_\mu), \quad \forall i.$$

Thus finding sections in $\Gamma_q^{\mathbf{k}}(V_\mu)$ is equivalent to finding, for all $\lambda \in \mathcal{P}_+$, the homomorphisms $\phi^{(\lambda)} \in \mathrm{Hom}_{\mathrm{U}_q(\mathbf{k})}(W(\lambda), V_\mu)$. Note that each such homomorphism $\phi^{(\lambda)}$ determines d_λ linearly independent sections:

$$\zeta_i^{(\lambda)} = \sum_j \phi^{(\lambda)}(w_j^{(\lambda)}) \otimes \tilde{t}_{ij}^{(\lambda)}.$$

However, when μ is not integral with respect to $\mathrm{U}_q(\mathbf{g})$, $\mathrm{Hom}_{\mathrm{U}_q(\mathbf{k})}(W(\lambda), V_\mu) = 0$, and hence $\Gamma_q^{\mathbf{k}}(V_\mu)$ vanishes in this case.

Now consider the case with $\mu = 0$; we have $\Gamma_q^{\mathbf{k}}(V_{\mu=0}) = \mathcal{A}_q^{\mathbf{k}}$ as supervector spaces. There is a homomorphism from the trivial representation of $\mathrm{U}_q(\mathbf{g})$, $W(0) = \mathbf{C}$, onto $V_0 = \mathbf{C}$. This gives the constant sections of $\mathcal{A}_q^{\mathbf{k}}$. Let γ be the highest root of $\mathbf{g}$. Recall that in the classical situation, $\mathbf{k}$ is reductive with $N = r - |\Theta|$ independent central elements. This, transcribed to the quantum case, implies the existence of N linearly independent $\mathrm{U}_q(\mathbf{k})$ homomorphisms $W(\gamma) \to \mathbf{C}$. As mentioned above, each of these corresponds to $d = \dim(\mathbf{g})$ linearly independent sections. So the representation $W(\gamma)$ determines Nd linearly independent sections. Further linearly independent sections can be obtained using the following lemma.

Lemma 2: Suppose there are nontrivial $\mathrm{U}_q(\mathbf{k})$ *homomorphisms* $W(\lambda_1) \to V_{\mu_1}$ *and* $W(\lambda_2) \to V_{\mu_2}$. *Then there is an induced nontrivial* $\mathrm{U}_q(\mathbf{k})$ *homomorphism*

$$W(\lambda_1 + \lambda_2) \to V_{\mu_1 + \mu_2}.$$

For example, for any positive integer m, there exist $(m|N)$ (partition of m into $\leqslant N$ parts) linearly independent homomorphisms $W(m\gamma)\rightarrow\mathbf{C}$. Thus we have proved that the algebra $\mathcal{A}_q^{\mathbf{k}}$ is infinite dimensional.

Now let us consider the case with $0\neq\mu\in\mathcal{P}$. It is an elementary exercise to verify that V_μ is $\mathrm{U}_q(\mathbf{k})$-isomorphic to a $\mathrm{U}_q(\mathbf{k})$-irreducible part of $W(\lambda')$, where λ' is the dominant weight in the Weyl group orbit of μ. Thus there is a nontrivial $\mathrm{U}_q(\mathbf{k})$ homomorphism

$$W(\lambda')\rightarrow V_\mu,$$

and this determines at least $d_{\lambda'}$ linearly independent sections in $\Gamma_q^{\mathbf{k}}(V_\mu)$. Further linearly independent sections can be constructed explicitly using Lemma 2 which promises a family of homomorphisms

$$W(\lambda'+m\gamma)\rightarrow V_\mu,\quad m\in\mathbf{N}_+.$$

This establishes that $\Gamma_q^{\mathbf{k}}(V_\mu)$ is infinite dimensional. □

$\mathcal{A}_q^{\mathbf{k}}$ may be regarded as the quantum analog of the algebra of functions over the superspace $\mathrm{OSP}(1|2n)/K$, where K is the subgroup of $\mathrm{OSP}(1|2n)$ with Lie superalgebra $\mathbf{k}$. Such homogeneous superspaces were studied in the work of Manin,[16] Penkov,[17] and others. Here we wish to make some investigations into their quantum analogs.

As is well known, one cannot define a noncommutative (in the $\mathbf{Z}_2$-graded sense) space directly in geometrical terms. Instead, such a space has to be defined by specifying its algebra of functions. We will take $\mathcal{A}_q^{\mathbf{k}}$ as the algebra of functions over the quantum homogeneous superspace which corresponds to $\mathrm{OSP}(1|2n)/K$ in the classical situation. Let us now study properties of $\Gamma_q^{\mathbf{k}}(V)$. First observe the following.

Theorem 3: *$\Gamma_q^{\mathbf{k}}(V)$ furnishes a two-sided $\mathcal{A}_q^{\mathbf{k}}$ module under the multiplication of $\mathcal{T}_q(\mathbf{g})$.*

Proof: The left and right actions of $\mathcal{A}_q^{\mathbf{k}}$ on $\Gamma_q^{\mathbf{k}}(V)$ are respectively defined by

$$a\zeta=\sum_r(-1)^{[a][v_i]}v_i\otimes af_i,\quad \zeta a=\sum_r v_i\otimes f_i a,$$

where $a\in\mathcal{A}_q^{\mathbf{k}}$ and $\zeta=\Sigma_i v_i\otimes f_i\in\Gamma_q^{\mathbf{k}}(V)$. Now for $p\in\mathrm{U}_q(\mathbf{k})$,

$$p\circ(a\zeta)=\sum_{(p)}(-1)^{[p_{(2)}][a]}\{p_{(1)}\circ a\}\{p_{(2)}\circ\zeta\}=(-1)^{[p][a]}a\{p\circ\zeta\}=(S(p)\otimes\mathrm{id}_{\mathcal{T}_q(\mathbf{g})})a\zeta;$$

$$p\circ(\zeta a)=\sum_{(p)}(-1)^{[p_{(2)}][\zeta]}\{p_{(1)}\circ\zeta\}\{p_{(2)}\circ a\}=\{p\circ\zeta\}a=(S(p)\otimes\mathrm{id}_{\mathcal{T}_q(\mathbf{g})})\zeta a.$$

This completes the proof. □

When V is actually a $\mathrm{U}_q(\mathbf{g})$ module, the $\mathcal{A}_q^{\mathbf{k}}$ module $\Gamma_q^{\mathbf{k}}(V)$ has a particularly simple structure.

Proposition 2: Let W be a finite-dimensional left $\mathrm{U}_q(\mathbf{g})$ module, which we regard as a left $\mathrm{U}_q(\mathbf{k})$ module by restriction. Then $\Gamma_q^{\mathbf{k}}(W)$ is isomorphic to $W\otimes\mathcal{A}_q^{\mathbf{k}}$ either as a left or right $\mathcal{A}_q^{\mathbf{k}}$ module.

Proof: We first construct the right $\mathcal{A}_q^{\mathbf{k}}$ module isomorphism. Being a left $\mathrm{U}_q(\mathbf{g})$ module, W carries a natural right $\mathcal{T}_q(\mathbf{g})$ comodule structure with the comodule action $\delta: W\rightarrow W\otimes\mathcal{T}_q(\mathbf{g})$ defined by

$$\delta(w)(x)=xw,\quad x\in\mathrm{U}_q(\mathbf{g}),\quad w\in W. \tag{12}$$

[Here the notation requires some clarification. If we express $\delta(w)=\Sigma_{(w)}w_{(1)}\otimes w_{(2)}$, then $\delta(w)(x)=\Sigma_{(w)}(-1)^{[x][w_{(1)}]}w_{(1)}\langle w_{(2)},x\rangle$.] Define $\eta: W\otimes\mathcal{T}_q(\mathbf{g})\rightarrow W\otimes\mathcal{T}_q(\mathbf{g})$ by the composition of maps

$$W\otimes\mathcal{T}_q(\mathbf{g}) \xrightarrow{\delta\otimes \mathrm{id}} W\otimes\mathcal{T}_q(\mathbf{g})\otimes\mathcal{T}_q(\mathbf{g})\rightarrow W\otimes\mathcal{T}_q(\mathbf{g}),$$

where the last map is the multiplication of $\mathcal{T}_q(\mathbf{g})$. Then η defines a right $\mathcal{A}_q^{\mathbf{k}}$ module isomorphism, with the inverse map given by the composition

$$W\otimes\mathcal{T}_q(\mathbf{g}) \xrightarrow{\delta\otimes \mathrm{id}} W\otimes\mathcal{T}_q(\mathbf{g})\otimes\mathcal{T}_q(\mathbf{g}) \xrightarrow{(\mathrm{id}\otimes S\otimes \mathrm{id})} W\otimes\mathcal{T}_q(\mathbf{g})\otimes\mathcal{T}_q(\mathbf{g})\rightarrow W\otimes\mathcal{T}_q(\mathbf{g}),$$

where the last map is again the multiplication of $\mathcal{T}_q(\mathbf{g})$. It is not difficult to show that

$$x\circ\eta(\zeta)=\eta\left(\sum_{(x)}(x_{(1)}\otimes \mathrm{id}_{\mathcal{T}_q(\mathbf{g})})x_{(2)}\circ\zeta\right),$$

$$x\circ\eta^{-1}(\zeta)=\eta^{-1}\left(\sum_{(x)}(S(x_{(1)})\otimes \mathrm{id}_{\mathcal{T}_q(\mathbf{g})})x_{(2)}\circ\zeta\right),\quad \forall\zeta\in\mathcal{T}_q(\mathbf{g})\otimes W,\quad x\in \mathrm{U}_q(\mathbf{g}).$$

Consider $\zeta\in\Gamma_q^{\mathbf{k}}(W)$. We have

$$p\circ\eta(\zeta)=\eta\left(\sum_{(p)}(p_{(1)}\otimes \mathrm{id}_{\mathcal{T}_q(\mathbf{g})})p_{(2)}\circ\zeta\right)=\eta\left(\sum_{(p)}(p_{(1)}S(p_{(2)})\otimes \mathrm{id}_{\mathcal{T}_q(\mathbf{g})})\zeta\right)$$
$$=\epsilon(p)\,\eta(\zeta),\quad \forall p\in \mathrm{U}_q(\mathbf{k}).$$

Hence $\eta(\Gamma_q^{\mathbf{k}}(W))\subset W\otimes\mathcal{A}_q^{\mathbf{k}}$. Conversely, given any $\xi\in W\otimes\mathcal{A}_q^{\mathbf{k}}$, we have

$$p\circ\eta^{-1}(\xi)=\eta^{-1}\left(\sum_{(p)}(S(p_{(1)})\otimes \mathrm{id}_{\mathcal{T}_q(\mathbf{g})})p_{(2)}\circ\xi\right)$$
$$=\eta^{-1}\left(\sum_{(p)}(S(p_{(1)})\epsilon(p_{(2)})\otimes \mathrm{id}_{\mathcal{T}_q(\mathbf{g})})\xi\right)=(S(p)\otimes \mathrm{id}_{\mathcal{T}_q(\mathbf{g})})\,\eta^{-1}(\xi),\quad \forall p\in \mathrm{U}_q(\mathbf{k}).$$

Thus $\eta^{-1}(W\otimes\mathcal{A}_q^{\mathbf{k}})\subset\Gamma_q^{\mathbf{k}}(W)$. Therefore the restriction of η to $\Gamma_q^{\mathbf{k}}(W)$ provides the desired right $\mathcal{A}_q^{\mathbf{k}}$ module isomorphism.

The left module isomorphism is given by the restriction to $\Gamma_q^{\mathbf{k}}(W)$ of the linear map $\kappa: W\otimes\mathcal{T}_q(\mathbf{g})\rightarrow W\otimes\mathcal{T}_q(\mathbf{g})$, which is defined by the following composition of maps

$$W\otimes\mathcal{T}_q(\mathbf{g}) \xrightarrow{\delta\otimes \mathrm{id}} W\otimes\mathcal{T}_q(\mathbf{g})\otimes\mathcal{T}_q(\mathbf{g}) \xrightarrow{\mathrm{id}\otimes P(S^2\otimes \mathrm{id})} W\otimes\mathcal{T}_q(\mathbf{g})\otimes\mathcal{T}_q(\mathbf{g})\rightarrow W\otimes\mathcal{T}_q(\mathbf{g}),$$

where

$$P:\mathcal{T}_q(\mathbf{g})\otimes\mathcal{T}_q(\mathbf{g})\rightarrow\mathcal{T}_q(\mathbf{g})\otimes\mathcal{T}_q(\mathbf{g}),$$
$$a\otimes b\mapsto(-1)^{[a][b]}b\otimes a. \tag{13}$$

The inverse map κ^{-1} is given by

$$W\otimes\mathcal{T}_q(\mathbf{g}) \xrightarrow{\delta\otimes \mathrm{id}} W\otimes\mathcal{T}_q(\mathbf{g})\otimes\mathcal{T}_q(\mathbf{g}) \xrightarrow{\mathrm{id}\otimes P(S\otimes \mathrm{id})} W\otimes\mathcal{T}_q(\mathbf{g})\otimes\mathcal{T}_q(\mathbf{g})\rightarrow W\otimes\mathcal{T}_q(\mathbf{g}).$$

□

With the help of this Proposition, we can now prove the following important result.

Theorem 4: $\Gamma_q^{\mathbf{k}}(V)$ *is projective and of finite type both as a left and right module over the superalgebra* $\mathcal{A}_q^{\mathbf{k}}$ *of functions on the quantum homogeneous superspace.*

Proof: Since $\mathrm{U}_q(\mathbf{k})$ is a reductive subalgebra of $\mathrm{U}_q(\mathbf{g})$, all finite-dimensional representations of $\mathrm{U}_q(\mathbf{k})$ are completely reducible. Let V_s, $s=1,2,...,K<\infty$, be the irreducible direct summands of

V such that their weights are all integral with respect to $\mathrm{U}_q(\mathbf{g})$. Then $\Gamma_q^{\mathbf{k}}(V)=\oplus_s\Gamma_q^{\mathbf{k}}(V_s)$. Consider any V_s, and denote its highest weight by μ_s. There exists such a $\hat{\mu}_s$ in the Weyl group orbit of $\mathbf{g}$ that is integral dominant with respect to $\mathbf{g}$. Let $W(\hat{\mu}_s)$ be the irreducible $\mathrm{U}_q(\mathbf{g})$ module with highest weight $\hat{\mu}_s$, which can be regarded as a $\mathrm{U}_q(\mathbf{k})$ module in the natural way. There always exists a $\mathrm{U}_q(\mathbf{k})$ module $V_s^{\perp}$ such that $W(\hat{\mu}_s)=V_s\oplus V_s^{\perp}$. Write $V^{\perp}=\oplus_s V_s^{\perp}$, and $W=\oplus_s W(\hat{\mu}_s)$. We have

$$\Gamma_q^{\mathbf{k}}(V)\oplus\Gamma_q^{\mathbf{k}}(V^{\perp})=\Gamma_q^{\mathbf{k}}(W)\cong W\otimes\mathcal{A}_q^{\mathbf{k}},$$

where the last step follows from Proposition 2. □

Recall that in classical differential geometry, the space $\mathcal{H}$ of sections of a vector bundle over a compact manifold M furnishes a module over the algebra $\mathcal{A}(M)$ of functions on M. It then follows from Swan's theorem that this module must be projective and is of finite type. Conversely, any projective module of finite type over $\mathcal{A}(M)$ is isomorphic to the space of sections of some vector bundle over M. This result is taken as the starting point for studying vector bundles in noncommutative geometry: one defines a vector bundle over a noncommutative space in terms of the space of sections which is required to be a finite-type project module over the noncommutative algebra of functions on the virtual noncommutative space. Therefore, $\Gamma_q^{\mathbf{k}}(V)$ will be called the space of sections of a quantum supervector bundle over the quantum homogeneous superspace associated with $\mathcal{A}_q^{\mathbf{k}}$.

Homogeneous supervector bundles at the classical level were studied in Refs. 16 and 17. We will not enter the discussion of the subject, but merely mention that the subject proves to be extremely rich and many aspects of it remain to be developed.

Following the classical terminology, we will call a quantum supervector bundle trivial if the sections form a free module over the superalgebra of functions on the quantum superspace. The following proposition is an immediate consequence of Proposition 2.

Proposition 3: If the $\mathrm{U}_q(\mathbf{k})$ *module V is in fact a finite-dimensional left* $\mathrm{U}_q(\mathbf{g})$ *module, then the quantum homogeneous supervector bundle with the space of sections* $\Gamma_q^{\mathbf{k}}(V)$ *is trivial.*

V. INDUCED REPRESENTATIONS

In this section we will investigate induced representations of the quantum supergroup $\mathrm{OSP}_q(1|2n)$ by using results of the last section. The following proposition explains how quantum homogeneous supervector bundles enter representation theory.

Proposition 4: $\Gamma_q^{\mathbf{k}}(V)$ *furnishes a left* $\mathrm{U}_q(\mathbf{g})$ *module under the* $\cdot$ *action, and also a right* $\mathcal{T}_q(\mathbf{g})$ *comodule under the action* $\omega=\mathrm{id}_V\otimes(\mathrm{id}_{\mathcal{T}_q(\mathbf{g})}\otimes S^{-1})\Delta$.

Proof: For $p\in\mathrm{U}_q(\mathbf{k})$, $x\in\mathrm{U}_q(\mathbf{g})$, and $\zeta\in\Gamma_q^{\mathbf{k}}(V)$, we have

$$p\circ(x\cdot\zeta)=(-1)^{[p][x]}x\cdot(p\circ\zeta)=(S(p)\otimes\mathrm{id}_{\mathcal{T}_q(\mathbf{g})})(x\cdot\zeta).$$

Thus $\Gamma_q^{\mathbf{k}}(V)$ indeed furnishes a left $\mathrm{U}_q(\mathbf{g})$ module under the $\cdot$ action. The $\mathcal{T}_q(\mathbf{g})$ coaction ω is just the dual of this left $\mathrm{U}_q(\mathbf{g})$ action.

We call $\Gamma_q^{\mathbf{k}}(V)$ an induced $\mathrm{U}_q(\mathbf{g})$ module, and also an induced $\mathcal{T}_q(\mathbf{g})$ comodule. For such induced modules, we have the following quantum analog of Frobenius reciprocity.

Theorem 5: *Let W be a* $\mathrm{U}_q(\mathbf{g})$ *module, the restriction of which furnishes a* $\mathrm{U}_q(\mathbf{k})$ *module in a natural way. Then there exists a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{U}_q(\mathbf{g})}(W,\Gamma_q^{\mathbf{k}}(V))\cong\mathrm{Hom}_{\mathrm{U}_q(\mathbf{k})}(W,V), \tag{14}$$

where $\mathrm{U}_q(\mathbf{g})$ *acts on the left module* $\Gamma_q^{\mathbf{k}}(V)$ *via the* $\cdot$ *action.*

Proof: We prove the proposition by explicitly constructing the isomorphism, which we claim to be the linear map

$$F:\mathrm{Hom}_{\mathrm{U}_q(\mathbf{g})}(W,\Gamma_q^{\mathbf{k}}(V))\to\mathrm{Hom}_{\mathrm{U}_q(\mathbf{k})}(W,V),\quad \psi\mapsto\psi(1_{\mathrm{U}_q(\mathbf{g})}),$$

with the inverse map

$$\overline{F}:\mathrm{Hom}_{\mathrm{U}_q(\mathbf{k})}(W,V)\to\mathrm{Hom}_{\mathrm{U}_q(\mathbf{g})}(W,\Gamma_q^{\mathbf{k}}(V)),\quad \phi\mapsto\overline{\phi}=(\phi\otimes S)\delta,$$

where $\delta:W\to W\otimes\mathcal{T}_q(\mathbf{g})$ is the right $\mathcal{T}_q(\mathbf{g})$ comodule action defined by (12).

To verify our claim, we first need to demonstrate that the image of F is contained in $\mathrm{Hom}_{\mathrm{U}_q(\mathbf{k})}(W,V)$. Consider $\psi\in\mathrm{Hom}_{\mathrm{U}_q(\mathbf{g})}(W,\Gamma_q^{\mathbf{k}}(V))$. For any $p\in\mathrm{U}_q(\mathbf{k})$ and $w\in W$, we have

$$p(F\psi(w))=(S^{-1}(p)\circ\psi(w))(\mathbb{1}_{\mathrm{U}_q(\mathbf{g})}),$$

where we have used the defining property of $\Gamma_q^{\mathbf{k}}(V)$. Note that

$$(S^{-1}(p)\circ\psi(w))(\mathbb{1}_{\mathrm{U}_q(\mathbf{g})})=(p\cdot\psi(w))(\mathbb{1}_{\mathrm{U}_q(\mathbf{g})}).$$

The $\mathrm{U}_q(\mathbf{g})$-module structure of $\Gamma_q^{\mathbf{k}}(V)$ and the given condition that ψ is a $\mathrm{U}_q(\mathbf{g})$-module homomorphism immediately leads to

$$p(F\psi(w))=(-1)^{[\psi][p]}\psi(pw)(\mathbb{1}_{\mathrm{U}_q(\mathbf{g})})=(-1)^{[\psi][p]}F\psi(pw),\quad p\in\mathrm{U}_q(\mathbf{k});\quad w\in W.$$

Now consider $\overline{F}$. We first show that the image $\mathrm{Im}(\overline{F})$ of $\overline{F}$ is contained in $\mathrm{Hom}_{\mathrm{U}_q(\mathbf{g})}(W,\Gamma_q^{\mathbf{k}}(V))$. Note that $\mathrm{Im}(\overline{F})\subset\mathrm{Hom}_{\mathbf{C}}(W,V\otimes\mathcal{T}_q(\mathbf{g}))$. Some relatively simple manipulations lead to

$$(x\cdot\overline{\phi}(w))=\overline{\phi}(xw),$$

$$(p\circ\overline{\phi}(w))=(S(p)\otimes\mathrm{id}_{\mathcal{T}_q(\mathbf{g})})\overline{\phi}(w),\quad x\in\mathrm{U}_q(\mathbf{g}),\quad p\in\mathrm{U}_q(\mathbf{k}),\quad w\in W.$$

Therefore, $\mathrm{Im}(\overline{F})\subset\mathrm{Hom}_{\mathrm{U}_q(\mathbf{g})}(W,\Gamma_q^{\mathbf{k}}(V))$. Now we show that F and $\overline{F}$ are inverse to each other. For $\psi\in\mathrm{Hom}_{\mathrm{U}_q(\mathbf{g})}(W,\Gamma_q^{\mathbf{k}}(V))$, and $\phi\in\mathrm{Hom}_{\mathrm{U}_q(\mathbf{k})}(W,V)$, we have

$$(F\overline{F}\phi)(w)=(\overline{F}\phi)(w)(1_{\mathrm{U}_q(\mathbf{g})})=\phi(w),$$

$$\begin{aligned}(\overline{F}F\psi)(w)(x)&=(-1)^{[x]([w]+1)}(F\psi)(S(x)w)\\&=(-1)^{[x]([w]+1)}\psi(S(x)w)(1_{\mathrm{U}_q(\mathbf{g})})\\&=(-1)^{[x]([w]+[\psi]+1)}(S(x)\cdot\psi(w))(1_{\mathrm{U}_q(\mathbf{g})})=\psi(w)(x),\quad x\in\mathrm{U}_q(\mathbf{g}),\quad w\in W.\end{aligned}$$

This completes the proof of the Proposition. □

Let V_μ be a finite-dimensional irreducible $\mathrm{U}_q(\mathbf{p})$ module with highest weight μ and lowest weight $\tilde{\mu}$. Since V_μ is a $\mathrm{U}_q(\mathbf{p})$ module the following is a well-defined subspace of $\Gamma_q^{\mathbf{k}}(V_\mu)$,

$$\mathcal{O}_q(V_\mu):=\{\zeta\in\Gamma_q^{\mathbf{k}}(V_\mu)|p\circ\zeta=(S(p)\otimes\mathrm{id}_{\mathcal{T}_q(\mathbf{g})})\zeta,\quad\forall p\in\mathrm{U}_q(\mathbf{p})\}.$$

We may regard $\mathcal{O}_q(V_\mu)$ as the quantum analog of the space of "holomorphic sections." Recall that the notation $W(\lambda)$ denotes the irreducible $\mathrm{U}_q(\mathbf{g})$ module with highest weight λ. We have the following result.

Theorem 6: *There exists the following* $\mathrm{U}_q(\mathbf{g})$ *module isomorphism*

$$\mathcal{O}_q(V_\mu)\cong\begin{cases}W((-\tilde{\mu})^{\dagger}), & -\tilde{\mu}\in\mathcal{P}_+,\\ 0, & \text{otherwise.}\end{cases}\tag{15}$$

Proof: Each $\zeta \in \mathcal{O}_q(V_\mu)$ can be expressed in the form

$$\zeta = \sum_{\lambda \in \mathcal{P}_+} \sum_{i,j} v_{ij}^{(\lambda)} \otimes \tilde{t}_{ij}^{(\lambda)},$$

for some $v_{ij}^{(\lambda)} \in V_\mu$ $(i,j=1,...,d_\lambda)$. Arguing as in the proof of Proposition 1 one concludes, for each $\lambda \in \mathcal{P}_+$, that there exist $\phi_i^{(\lambda)} \in \mathrm{Hom}_{\mathbf{C}}(W(\lambda), V_\mu)$ such that $v_{ij}^{(\lambda)} = \phi_i^{(\lambda)}(w_j^{(\lambda)})$, where $\{w_i^{(\lambda)}\}$ is the basis of $W(\lambda)$, relative to which the irreducible representation $t^{(\lambda)}$ of $\mathrm{U}_q(\mathbf{g})$ is defined. Thus we can rewrite ζ as

$$\zeta = \sum_{\lambda \in \mathcal{P}_+} \sum_{i,j} \phi_i^{(\lambda)}(w_j^{(\lambda)}) \otimes \tilde{t}_{ij}^{(\lambda)}.$$

Similar reasoning as in the proof of Proposition 1 shows that the $\phi_i^{(\lambda)}$ must be $\mathrm{U}_q(\mathbf{p})$-module homomorphisms of degree $[\phi_i^{(\lambda)}]$. It immediately follows from (4) that

$$\phi_i^{(\lambda)} = c_i \phi^{(\lambda)}, \quad c_i \in \mathbf{C},$$

and $\phi^{(\lambda)}$ may be nonzero only when

$$\overline{\lambda} = \tilde{\mu}.$$

Hence, if $-\tilde{\mu} \notin \mathcal{P}_+$, we have $\mathcal{O}_q(V_\mu) = 0$. When $-\tilde{\mu} \in \mathcal{P}_+$, we set

$$\nu = (-\tilde{\mu})^\dagger.$$

Then, we may conclude that $\mathcal{O}_q(V_\mu)$ is spanned by

$$\zeta_i = \sum_j \phi^{(\nu)}(w_j^{(\nu)}) \otimes \tilde{t}_{ij}^{(\lambda)}, \tag{16}$$

which are obviously linearly independent. Furthermore,

$$x \cdot \zeta_i = (-1)^{[x][\phi^{(\nu)}]} \sum_j t_{ji}^{(\nu)}(x) \zeta_j, \quad x \in \mathrm{U}_q(\mathbf{g}).$$

Thus $\mathcal{O}_q(V_\mu) \cong W(\nu)$. More explicitly, the isomorphism is given by

$$W(\nu) \xrightarrow{(\mathrm{id} \otimes S)\delta} \mathcal{O}_q(W(\nu)) \xrightarrow{\phi^{(\nu)} \otimes \mathrm{id}} \mathcal{O}_q(V_\mu). \tag{17}$$

This completes the proof of the theorem. □

This result provides an analog of the celebrated Borel–Weil theorem for the quantum supergroup $\mathrm{OSP}_q(1|2n)$. For the classical Lie supergroups, the program of developing a Bott–Borel–Weil theory was extensively investigated by Penkov and co-workers.[17] Also, a quantum Borel–Weil theorem for the covariant and contravariant tensor representations of quantum $\mathrm{GL}(m|n)$ was obtained in Ref. 9.

When $\mu = 0$, the theorem implies that

$$\{f \in \mathcal{T}_q(\mathbf{g}) | p \circ f = \epsilon(p) f, \quad \forall p \in \mathrm{U}_q(\mathbf{p})\} = \mathbf{C}\epsilon.$$

Combining this result with with Proposition 2, we obtain the following

Corollary: Let W be any finite-dimensional $\mathrm{U}_q(\mathbf{g})$ *module. Then, as* $\mathrm{U}_q(\mathbf{g})$*-modules,*

$$\mathcal{O}_q(W) \cong \epsilon \otimes W.$$

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